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On Cardinal Numbers.

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Preface.

The following memoir is concerned solely with the theory of Cardinal numbers, and does not touch upon the theory of ordered aggregates or of ordinal types which form the other parts of the general theory of aggregates (mengenlehre).*

Sections I and II are preliminary, to explain the notation and methods of reasoning. Incidentally they may be of independent service as explaining the elements of Peano's developments of mathematical logic† (Section I) and of Russell's symbolism for the Logic of Relations‡ (Section II). I believe that these two methods are almost indispensable for the development of the theory of Cardinal numbers. The abstract nature of the subject makes ordinary language totally ineffective, only gaining precision by verbosity, and imagination is very misleading, since it presents to us special aggregates which are denumerable or of the power of the continuum. Thus we are thrown back onto a strict logical deduction by a symbolical method. I believe that the invention of the Peano and Russell symbolism, used here, forms an epoch in mathematical reasoning.

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* Due almost entirely to G. Cantor. Almost all that is known respecting Cardinal Numbers, apart from their relations to the other parts of the theory, is given in his memoir in Math. Annal., Bd. XLVI, 1895 : this and its sequel on Ordinal Types, loc. cit. Bd. XLIX, can be obtained as one pamphlet in a French translation, translated by F. Marotte, Paris, A. Hermann, 1899. For a general sketch of the whole theory, with bibliography, cf. "Die Entwickelung der Lehre von den Punktmannigfaltigkeiten" : Bericht von A. Schönflies, "Yahresbericht der Deutschen Mathematiker-Vereinigung," 1900. For a development of the theory of Ordinal Types in respect to condensed series, by the methods of the present memoir, cf. Bertrand Russell, Revue de Mathématiques, vol. VII, Turin, 1901, where the methods of reasoning used here were first elaborated.

† For references to Peano's articles, cf. my memoir on the "Algebra of Symbolic Logic," in this Journal, vol. XXIII.

‡ Cf. Russell, loc. cit.

Section III is entirely due to Russell, and is written by him throughout. It deals with the principles of the subject, and effects a considerable simplification in them. Two classes are defined [cf. ★ 1·1 (all references are to the series of propositions commencing with Section III and thence numbered consecutively)] as similar when a one-one relation can be established between their numbers. The cardinal number (or power) of a class u is defined simply as the class of those classes similar to u (cf. ★ 1·3); thus, for instance, the cardinal number 2 is the class of those classes which (in ordinary language) contain two members, and we shall say that a class containing two members is a 2. The finite cardinals are defined [cf. ★ 1·4 to ★ 1·61] as those cardinal numbers which can be obtained by induction.* The class of infinite cardinals is defined [cf. ★ 1·8] as the class of cardinals which are not finite. The whole class of cardinal numbers is denoted by Nc. The properties of finite and infinite cardinals are deduced by strict logical reasoning from these definitions. In particular, if the cardinal number of the class of finite cardinals is denoted by α_0, it is proved that every infinite class contains a class whose cardinal number is α_0 [cf. ★ 2·6], and that no finite class is similar to any part of itself [cf. ★ 2·75], and that every infinite class is similar to some part of itself [cf. ★ 2·76]. The cardinal and ordinal theories of finite number are connected in ★ 3·3.

The known properties of α_0 are proved [★ 4·1 to ★ 4·24]. The theorems that if α and β are *any* two cardinals, and α is infinite and $\alpha \geqq \beta$, then

$$\alpha + \beta = \alpha \quad \text{and} \quad \alpha \times \beta = \alpha:$$

cannot be deduced from these premises, and, as far as I am aware, have never been proved. Russell points out that the first of them follows from another unproved theorem [★ 4·3]. The class of numbers for which the first theorem holds is considered (cf. ★ 30). The second of the theorems cannot be thus deduced; but in Section V, ★ 20·1, it is proved that if α belong to the class for which the first theorem holds and $\beta \geqq \delta$, then

$$\beta^\alpha \times \delta^\alpha = \beta^\alpha.$$

The similar cases for $\alpha + \beta = \alpha$ are examined in ★ 30·0 to ★ 31·3.

Russell also gives exact proofs of such theorems as, $\beta < \gamma$ implies

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* The recognition of the importance of Induction is due to Dedekind, cf. "Was sind and was sollen die Zahlen ?" 1887.

$\alpha+\beta\leqq\alpha+\gamma$, and $\alpha\times\beta\leqq\alpha\times\gamma$, and $\alpha^\beta\leqq\alpha^\gamma$, and $\beta^\alpha\leqq\gamma^\alpha$ [cf. ★4·37, ★4·39, ★4·4, ★4·41].

Also ★4·5 contains another proof (assuming ★ 4·3) of Bernstein's and Schröder's theorem, that $\alpha\geqq\beta$ and $\beta\geqq\alpha$ imply $\alpha=\beta$.

Throughout Sections IV and V, I am indebted to Russell for great improvements in the symbolism of the proofs and for the correction of many oversights. Section IV is devoted to the general consideration of addition and multiplication when the number of summands or of factors is not necessarily finite or even denumerable. Cantor's definitions only apply to these cases [cf. ★7·0 to ★7·21]. The associative laws are proved [cf. ★8·1 to ★8·31, and ★10·0 to ★10·22] both for addition and multiplication when the number of brackets, as well as the number of numbers in each bracket, is not necessarily finite or denumerable. With these definitions the commutative laws become entirely irrelevant. The ordinary connection between addition and multiplication, exemplified in $\alpha+\alpha=2\times\alpha$, is proved [cf. ★7·3] to hold generally; also the distributive law [cf. ★11·0]. These definitions and theorems require several preliminary definitions; thus for instance [★5·1], that of a "class of classes exclusive," and [★6·0] of the "multiplicative class" of a class of classes exclusive; this last conception assumes great importance in the sequel; also, the general definitions of sums and products depend on the definition [★8·22] of "a class of classes of classes arithmetical." The remainder of Section IV is devoted to theorems following from these definitions.

Section V opens with the definition [★12·1] of a power (β^α); this definition follows from that of a product; and [★14·1] Cantor's definition is deduced from it. Thus, by this procedure, the ordinary relations between addition, multiplication, and powers of numbers are preserved in their full generality. The theory of combinations is then extended to infinite cardinals; thus, if α, β are numbers, and $\alpha\geqq\beta$, C^α_β is defined [cf. ★15·1 and ★16·0]: the extension of Vandermonde's theorem is proved [cf. ★16·1]; if α is infinite and such that $\alpha+\alpha=\alpha$, $C^\alpha_\alpha=2^\alpha$ [cf. ★16·23]: extensions of the binomial theorem are given in ★17·0 to ★19·7; it is proved [cf. ★17·4] that if α and β are any cardinals,

$$(1+\beta)^\alpha=\sum^{\nu\leqq\alpha} C^\alpha_\nu\times\beta^\nu.$$

If α is infinite and ν finite [cf. ★20·4], then $\nu^\alpha=2^\alpha$, and when α is infinite and such that $\alpha^2=\alpha$, then [cf. ★32·4], if $\beta\leqq 2^\alpha$, we have $\beta^\alpha=2^\alpha$.

The idea of the number of permutations (P^α_β) of α things, β at a time, is extended [cf. ★22·0 and ★22·01] to the case when α and β are infinite: then [cf. ★22·1] ${}^\beta P_\beta \times C^\alpha_\beta = P^\alpha_\beta$; also, if α is infinite, and $\alpha \geqq \beta$ and with certain other limitations, then [cf. ★22·2]

$$C^{\alpha\times\beta}_\beta = P^{\alpha\times\beta}_\beta = \alpha^\beta.$$

Hence [cf. ★22·21], if $\alpha \times \beta = \alpha$, we have

$$C^\alpha_\beta = P^\alpha_\beta = \alpha^\beta.$$

In ★30·0 to ★32·6 certain classes of numbers which, without assuming ★4·3, satisfy $\beta + \gamma = \beta$ and $\beta \times \gamma = \beta$ when $\beta \geqq \gamma$ are considered.

The Italian system of numeration of propositions, on the left-hand margin, by integral and decimal numbers, is adopted. It is unnecessary for me to emphasise further, what is evident from inspection, how largely this article is based upon the ideas of G. Cantor, and of Peano, and of Russell.

SECTION I.

Account of Peano's Logical Symbolism.

Logical addition and multiplication are denoted by $\cup$ and by $\cap$ instead* of by $+$ and by $\times$; when these signs connect propositions, they indicate, as usual, that the propositions are to be taken disjunctively or conjunctively. In this memoir simple juxtaposition of terms will never be used to indicate the logical multiplication of classes; but the multiplication of propositions (which is symbolised by $\cap$) will also be indicated by separating them by one dot, thus $a\,.\,b$, or two dots, thus $a:b$, or three dots, thus $a\,.\!\cdot\!.\,b$, or four dots, thus $a::b$; it is to be noted that $.\!\cdot\!.$ never stands for therefore.

A conjunctive or disjunctive complex of many propositions is itself one proposition. The different arrangements of dots in multiplication are therefore useful in the place of brackets.

The symbol $\supset$ between classes means "is contained in," and between propositions means "implies"; it is the same in signification as Schröder's $\mathrel{=\!\!\!\!(}$. Thus, if the letters denote propositions,

$$a \supset b\,.\,x\,.\,y :\supset .\, u \cup v \,.\!\cdot\!.\, \supset .\, w,$$

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* For example, cf. my previous memoir "On Symbolic Logic" in this Journal, vol. XXIII, No. 2.

is analysable into the hypothesis

$$a \supset b \,.\, x \,.\, y : \supset . \, u \cup v,$$

and into the protasis w. That symbol $\supset$, in a given chain of symbols, which is preceded or succeeded by a greater number of dots than any other symbol $\supset$ in the chain, is the main logical copula of the proposition connecting the hypothesis and the protasis: and the proposition is that the hypothesis implies the protasis.

In the above example, the hypothesis is as follows:

The proposition that the proposition a implies the proposition b, together with the propositions x and y, implies the proposition u or v or both. Thus the hypothesis is itself separable into a hypothesis and protasis. In other examples the same may be true of the protasis. In the above example, if a and b represent classes, the hypothesis becomes,

The proposition that the class a is contained in the class b, together with the propositions x and y, implies the proposition u or v or both.

The classes contained in a given class A are never to be confused with the entities which form the extension of the class A. The fact that x is a member of the class A is denoted by

$$x \,\varepsilon\, A.$$

Here ε means "is a member of", or simply "is a"; ε is chosen for this signification as being the initial letter of εστι.

Thus $y \supset A \,.\, x \,\varepsilon\, A$ may coexist with $x \,\varepsilon\, y$, but not in general with $x \supset y$ or with $x = y$, though exceptional cases arise in which $x \,\varepsilon\, y$ and $x \supset y$ coexist. As an example, let N be the class "Finite Integer", and Np the class "Prime Number", we have

$$2 \,\varepsilon\, N \,.\, Np \supset N \,.\; 2 \,\varepsilon\, Np$$

But 2 is a particular number and not a class of numbers; hence the symbolism, $2 \supset N$, is wrong.

Accordingly, $x \supset y$ implies either that x and y are both classes or that they are both propositions; and in this memoir single letters are not required to denote propositions.

If x and y are both members of the class A, then $x, y \,\varepsilon\, A$ is an abbreviation for $x \,\varepsilon\, A \,.\, y \,\varepsilon\, A$.

There are classes containing only one single member as their extension. If A is such a class, then ιA is used to denote the single member of the class A.

Also, conversely, if x is any entity, ιx is the class of which x is the single member. Hence, in symbols,

$$y \,\varepsilon\, \iota x \,.\supset.\, y = x \quad \text{and} \quad y = x \,.\supset.\, y \,\varepsilon\, \iota x,$$

and these two propositions are conveniently combined into

$$y \,\varepsilon\, \iota x \,.=.\, y = x;$$

thus the combination $\varepsilon\iota$ between classes is equivalent in signification to $=$.

For instance, the class of even primes contains the single member 2. If A denote this class, then

$$\iota 2 = A \,.\, 2 \,\varepsilon\, A \,.\, 2 = ℩A : x \,\varepsilon\, A \,.\supset.\, x = 2.$$

We shall have occasion [★11·0] to use the symbol $\iota\iota x$. This signifies the class whose single member is the class whose single member is x.

Again, if u is a class, $u \cup \iota x$ is the class whose members are the members of u together with the entity x. The symbolism $u \cup x$ for this meaning would have been wrong, for $\cup$ can only connect classes to form a new class whose extension is the joint extension of the two classes. Thus, if u and x are both classes, the two symbols $u \cup \iota x$ and $u \cup x$ have different meanings; we have

$$x \,\varepsilon\, u \cup \iota x \,.\, x \supset u \cup x;$$

also

$$y \,\varepsilon\, x \,.\supset.\, y \,\varepsilon\, u \cup x,$$

but $y \,\varepsilon\, x$ does not imply $y \,\varepsilon\, u \cup \iota x$.

The symbol elm, short for "element," stands for that class whose extension is formed by all classes whose extension is some single member. Thus, elm is a class whose members are themselves classes, but each containing only one member. Thus,

$$A \,\varepsilon\, \text{elm} \,.\supset: x \,\varepsilon\, A \,.\, y \,\varepsilon\, A \,.\supset.\, x = y :\supset.\, ℩A \,\varepsilon\, A.$$

The symbol Cls, short for Class, stands for that class whose extension is formed by all classes. Thus $x \,\varepsilon\, \text{Cls}$ simply means x is a class. We get here an exception to the rule that $x \,\varepsilon\, A$ is inconsistent with $x \supset A$, for Cls is itself a class, hence,

$$\text{Cls} \,\varepsilon\, \text{Cls} \,.\, \text{Cls} = \text{Cls}.$$

If p_x denotes a proposition containing a letter x which may be given various significations consistent with the truth of the proposition, then

$$x \ni (p_x)$$

means the class whose extension is all the entities x consistent with the truth of p_x. Thus $\ni$ may be read "such that." Thus $x \ni (x^2 = 1)$ is the class whose extension is formed by ± 1, and $x \ni (x = 2)$ is the class $\iota\, 2$. The symbol $x \ni (p_x)$ is always to be treated as one whole: thus, if A is a class, $A \cap x \ni (p_x)$ means the class common to A and to the class $x \ni (p_x)$. Thus,

$$A \cap x \ni (p_x) \,.\!=\!.\, x \ni (x \,\varepsilon\, A \,.\, p_x).$$

This symbol is especially useful in definitions, thus, for ιx, we have

$$\iota x = y \ni (y = x),$$

and for elm we have

$$\text{elm} = \text{cls} \cap x \ni (y,\, z \,\varepsilon\, x \,.\!\supset\!.\, y = z).$$

If p and q are propositions containing a variable x, then

$$p \,.\!\supset_x\!.\, q$$

means that p implies q for every value of x satisfying p: this subscript to $\supset$ is not required for the main implication of a proposition, and can sometimes be omitted when it is strictly required. Similarly, if p and q are propositions containing the variables x and y, then

$$p \,.\!\supset_{x,\, y}\!.\, q$$

means that p implies q for every value of x and y satisfying p.

Λ is the symbol for the null class: thus,

$$A \,\varepsilon\, \text{cls} \,.\!\supset_A\!.\, \Lambda \supset A.$$

If p is a proposition, $p = \Lambda$ is the proposition which is the denial of the proposition p.

If A is a class, $\exists A$ is the proposition that "A is not Λ": thus,

$$A \,\varepsilon\, \text{cls} \,.\!\supset\!:\, \exists A \,.\!=\!.\, (A = \Lambda) = \Lambda.$$

The legitimacy of $x \,\varepsilon\, A$ implies $\exists A$; thus, from the definition $u = x \ni (p_x)$, we cannot derive the legitimacy of some proposition $x \,\varepsilon\, u$ without the premise $\exists u$.

If A is a class, $-A$ is the class of all entities which are not A; if A is a proposition, $-A$ is the proposition which is the denial of A; also $-\varepsilon$ means "is not a member of", and $-\supset$ means "is not contained in"; thus,

$$A \,\varepsilon\, \text{cls} \,.\!\supset\!.\, -A = x \ni (x \,-\varepsilon\, A) = x \ni [(x \,\varepsilon\, A) = \Lambda] = x \ni [-(x \,\varepsilon\, A)],$$

and

$$A,\, B \,\varepsilon\, \text{cls} \,.\, A -\!\supset B \,.\!\supset\!.\, \exists A - B.$$

The symbol $\cap$ between classes is replaced in this memoir by simple juxtaposition in the sole case when $\cap$ immediately precedes the symbol $-$; thus,

$$A,\ B \varepsilon \mathrm{cls} \,.\supset.\, A - B = A \cap - B.$$

Also thus, if u is a class and $x \varepsilon u$, then $u - \iota x$ is the class u with the exception of the entity x. Note that $u - x$ for this meaning is wrong.

If A is a class, cls'A means the class whose members are the classes contained in A; thus,

$$A \varepsilon \mathrm{cls} \,.\supset.\, \mathrm{cls}'A = \mathrm{cls} \cap x \ni (x \supset A);$$

we might also have written

$$A \varepsilon \mathrm{cls} \,.\supset.\, \mathrm{cls}'A = x \ni (x \supset A),$$

leaving it to be inferred from the use of $\supset$ in $x \supset A$, that $x \varepsilon \mathrm{cls}$; thus,

$$\mathrm{cls}'\mathrm{cls} = \mathrm{cls} \cap x \ni (x \supset \mathrm{cls});$$

accordingly, cls'cls means the complete class of those classes whose members are themselves classes; thus,

$$A \varepsilon \mathrm{cls}'\mathrm{cls} \,.\, x \varepsilon A \,.\supset.\, x \varepsilon \mathrm{cls}.$$

Similarly, $\quad \mathrm{cls}'\,\mathrm{cls}'\mathrm{cls} = \mathrm{cls} \cap x \ni (x \supset \mathrm{cls}'\mathrm{cls}),$

and thus $\quad A \varepsilon\ \mathrm{cls}'\,\mathrm{cls}'\,\mathrm{cls} \,.\, x \varepsilon A \,.\supset.\, x \varepsilon \mathrm{cls}'\mathrm{cls}.$

These conceptions are essential in the general consideration of cardinal numbers. We shall abbreviate cls'cls into cls^2 and cls'cls'cls into cls^3.

We may note that

$$\mathrm{cls}^2 \varepsilon \mathrm{cls} \quad \text{and} \quad \mathrm{cls}^2 \supset \mathrm{cls};$$

also that

$$\mathrm{cls}^3 \varepsilon \mathrm{cls}^2, \text{ and } \mathrm{cls}^3 \varepsilon \mathrm{cls}, \text{ and } \mathrm{cls}^3 \supset \mathrm{cls}^2, \text{ and } \mathrm{cls}^3 \supset \mathrm{cls},$$

$$A \varepsilon \mathrm{cls}'\mathrm{cls} \,.\supset.\, \cup\text{‘}A = x \ni [\exists A \cap P \ni (x \varepsilon P)],$$

$$A \varepsilon \mathrm{cls}'\,\mathrm{cls} \,.\supset.\, \cap\text{‘}A = x \ni [P \varepsilon A \,.\supset_p.\, x \varepsilon P].$$

The class $\cup\text{‘}A$ is thus the logical sum of the classes which form the extension of A; thus, if the extension of A consists of only two classes B and C, then $\cup\text{‘}A = B \cup C$. The class $\cap\text{‘}A$ is the logical product of the classes which form the extension of A; thus, if the extension of A consists only of the two classes, B and C, then $\cap\text{‘}A = B \cap C$.

We shall, following Russell, employ frequently the following notation:

$x \varepsilon^2 A$ means that A is a cls' cls, and $x \varepsilon B$ where $B \varepsilon A$: thus in symbols,

$$A \varepsilon \mathrm{cls}'\mathrm{cls} \,.\supset:\, x \varepsilon^2 A \,.=.\, \exists A \cap B \ni (x \varepsilon B).$$

Note that $\exists A \cap B \ni (x \,\varepsilon\, B)$ affirms the existence of a class of B's such that $B \,\varepsilon\, A \,.\, x \,\varepsilon\, B$; it is obvious that to affirm the existence of a class of entities is the same as affirming the existence of at least one of them.

But this use of ε^2 brings us to Russell's "Logic of Relations", which we will now proceed to explain, shortening the explanations by the use of Peano's symbolism and numbering the definitions and propositions for reference. Df. (short for Definition) will be written after a definition, and Pp (short for Primitive proposition) after a proposition assumed without proof (cf. ★4·3 of Section III).

SECTION II.

Russell's "Logic of Relations."

Russell's* "Logic of Relations" may conveniently be considered as a generalization of the ideas suggested by Peano's proposition $x \,\varepsilon\, A$. This proposition asserts a special relation between x and A, namely, that A is a class and x is a member of its extension. Thus, according to Russell, if R is any relation, xRy signifies that x has the relation R to y. Let Rel. denote the complete class whose members are relations, then

1·00 $$\mathrm{R} \,\varepsilon\, \mathrm{Rel.} \supset : xRy \,.=.\, x \text{ has the relation } R \text{ to } y. \qquad \text{Df.}$$

Relations will always be denoted by capital letters to which there is a corresponding small Greek letter, and this Greek letter will have a fixed meaning, explained in 1·01, in relation to the corresponding capital italic letter. Small Greek letters will also be used in this memoir to represent cardinal numbers, but not those letters whose corresponding capital italics have been used for relations. The letters R, S will be confined to relations.

1·01 $$R \,\varepsilon\, \mathrm{rel.} \supset . \rho = x \ni [\exists y \ni (xRy)], \qquad \text{Df.}$$

1·02 $$R \,\varepsilon\, \mathrm{rel.} \supset . \breve{\rho} = y \ni [\exists x \ni (xRy)]. \qquad \text{Df.}$$

Here $\breve{\rho}$ is read as "ρ converse". Then ρ is the class of entities such that $x \,\varepsilon\, \rho$ implies that a class of entities (and thus at least one entity) such as y exists, such that x has the relation R to y. Similarly for $\breve{\rho}$. Here ρ is called the domain of R and $\breve{\rho}$ the converse domain of R.

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[*] Cf. Revue de Mathématiques (Turin), loc. cit.

1·03 $R \,\varepsilon\, \mathrm{rel} \,.\, x \,\varepsilon\, \rho \,.\supset.\, \breve{\rho}x = y \ni (xRy),$ Df.

1·04 $R \,\varepsilon\, \mathrm{rel} \,.\, y \,\varepsilon\, \breve{\rho} \,.\supset.\, \rho y = x \ni (xRy).$ Df.

Here $\breve{\rho}x$ and ρy are each to be considered as one symbol, and x and y are of the nature of indices.

1·05 $u \,\varepsilon\, \mathrm{cls} \,.\, R \,\varepsilon\, \mathrm{rel} \,.\, u \supset \rho \,.\supset.\, \breve{\rho}u = y \ni [\exists u \cap x \ni (xRy)],$ Df.

1·06 $u \,\varepsilon\, \mathrm{cls} \,.\, R \,\varepsilon\, \mathrm{rel} \,.\, u \supset \breve{\rho} \,.\supset.\, \rho u = x \ni [\exists u \cap y \ni (xRy)],$ Df.

1·10 $R,\, R' \,\varepsilon\, \mathrm{rel} \,.\supset.\because R \supset R' . \quad = : xRy \,.\supset_{x,\,y}.\, xR'y.$ Df.

This defines the meaning of "the relation R is contained in the relation R'"; thus, relations have some properties analogous to those of classes.

1·11 $R,\, R' \,\varepsilon\, \mathrm{rel} \,.\supset.\because R = R' . = : R \supset R' . R' \supset R.$ Df.

This defines the equivalence of relations.

1·12 $R \,\varepsilon\, \mathrm{rel} \,.\supset.\, \breve{R} = \iota\, \mathrm{rel} \cap R' \ni (xR'y \,.=.\, yRx),$ Df.

$\breve{R}$ is the converse relation to R.

1·13 $R \,\varepsilon\, \mathrm{rel} \,.\supset.\, \breve{\breve{R}} = R,$

1·20 $R_1,\, R_2 \,\varepsilon\, \mathrm{rel} \,.\supset: x (R_1 \cup R_2) y \,.=.\, xR_1y \,.\cup.\, xR_2y.$ Df.

This defines the "logical sum" of two relations; the logical sum $R_1 \cup R_2$ is itself a relation.

1·21 $R_1,\, R_2 \,\varepsilon\, \mathrm{rel} \,.\supset: x (R_1 \cap R_2) y \,.=.\, xR_1y \,.\, xR_2y.$ Df.

This defines the "logical product" of two relations; the logical product $R_1 \cap R_2$ is itself a relation.

1·22 $R_1 .\, R_2 \,\varepsilon\, \mathrm{rel} \,.\supset: xR_1R_2z \,.=.\, \exists y \ni (xR_1y \,.\, yR_2z).$ Df.

This defines the "relational product" of two relations; the relational product. R_1R_2 is itself a relation.

1·23 $K \,\varepsilon\, \mathrm{cls}'\mathrm{rel} \,.\supset: x \cup \text{‘}Ky \,.=.\, \exists K \cap R \ni (xRy).$ Df.

$\cup \text{‘}K$ is a relation; it is the "logical sum" of the relations forming the extension of the class K; this definition agrees with 1·20 when the extension of K is formed by two relations only.

1·30 $R \,\varepsilon\, \mathrm{rel} \,.\supset.\, R^2 = RR,$ Df.

1·31 $R,\, S \,\varepsilon\, \mathrm{rel} \,.\supset.\, {}^{\smile}(RS) = \breve{S}\breve{R},$

1·40 $R \,\varepsilon\, \mathrm{rel} \,.\supset: x - Ry \,.=.\, -(xRy).$ Df.

Thus $-R$ is the relation of not having the relation R.

2·01 $\varepsilon\varepsilon\ \mathrm{rel}$, (ε is a relation).

2·02 $x\,\varepsilon\,\breve{\varepsilon}\,y\,.=.\,\exists z\ni(x\,\varepsilon\,z\,.\,y\,\varepsilon\,z)$,

2·03 $x\,\breve{\varepsilon}\,\varepsilon\,y\,.=.\,\exists z\ni(z\,\varepsilon\,x\,.\,z\,\varepsilon\,y)\,.=.\,\exists xy$.

2·04 $y\,\varepsilon\,\mathrm{cls}\,'\,\mathrm{cls}\,.\supset.\,\cup\,'y=x\ni(x\,\varepsilon^2\,y)$,

2·11 $1'=\mathrm{identity}$. Df.

1' is the relation of identity; thus, $x1'x$; and $a\,\varepsilon\,\mathrm{elm}\,.\,x,\ y\,\varepsilon\,A\,.\supset.\,x1'y$.

2.12 $0'=-1'$. Df.

Thus 0' is the relation of diversity, that is, of "being different from".

2·13 $R,\ P\,\varepsilon\,\mathrm{rel}\,.\supset:R\breve{P}\supset 0'\,.=.\,\breve{R}P\supset 0'$.

3·01 $Nc\rightarrow 1\,.=.\,\mathrm{rel}\cap R\ni\{xRy\,.\,xRz\,.\supset_x.\,y1'z\}$. Df.

$Nc\rightarrow 1$ is the symbol for a certain class of relations, namely, the class of "many-one" relations; any x in the domain of R has the relation R to only one member of the converse domain.

3·02 $1\rightarrow Nc\,.=.\,\mathrm{rel}\cap R\ni\{yRx\,.\,zRx\,.\supset x\,.\,y1'z\}$. Df.

Thus $1\rightarrow Nc$ is the class of one-many relations.

3·03 $R\,\varepsilon\,Nc\rightarrow 1\,.=.\,\breve{R}\,\varepsilon\,1\rightarrow Nc$,

3·04 $R\,\varepsilon\,1\rightarrow Nc\,.\supset.\,R\breve{R}\supset 1'$,

that is, $xRy\,.\,y\breve{R}z\,.\supset.\,x1'z$.

3·11 $1\rightarrow 1\,.=.\,(Nc\rightarrow 1)\cap(1\rightarrow Nc)$.

$1\rightarrow 1$ is the class of one-one relations.

3·12 $R\,\varepsilon\,1\rightarrow 1\,.\supset.\,\breve{R}\,\varepsilon\,1\rightarrow 1$.

3·13 $R,\ S\,\varepsilon\,1\rightarrow 1\,.\supset.\,RS\,\varepsilon\,1\rightarrow 1$,

4·11 $S\,\varepsilon\,Nc\rightarrow 1\,.\,R=S\breve{S}\,.\supset.\,R^2\supset R\,.\,R=\breve{R}$.

4·12 $R\,\varepsilon\,\mathrm{rel}\,.\,R^2\supset R\,.\,R=\breve{R}\,.\,\exists R\,.\supset.\,\exists Nc\rightarrow 1\cap S\ni(R=S\breve{S})$.

Note that $R^2\supset R\,.=:xRy\,.\,yRz\,.\supset.\,xRz$,

and that $R=\breve{R}\,.=:xRy\,.=.\,yRx$.

5·11 $R\,\varepsilon\,\mathrm{rel}\,.\,u\supset\rho\,.\supset:xR_uy\,.=.\,x\,\varepsilon\,u\,.\,xRy$. Df.

5·12 $R\,\varepsilon\,\mathrm{rel}\,.\,u\supset\rho\,.\supset.\,\rho_u=u\,.\,\breve{\rho}_u=\breve{\rho}u$.

R_u is the relation R with its field limited, namely, the domain is limited to u, and thus the converse domain is limited to $\breve{\rho}u$.

For the rest of this memoir the propositions are numbered progressively onwards, and all references to propositions without indication of section refer to this series of numbers. In the future it will occur that a general hypothesis applies to a number of propositions: such a hypothesis will be stated only once, after mention of the integral number belonging to that set of propositions, and will be understood to hold till the integral number is changed. Demonstrations are placed in square brackets after the enunciations; subsidiary propositions used only in the demonstrations are numbered on the right instead of on the left, and only integral numbers in brackets are used for such numeration; thus, (2) would refer to a subsidiary proposition used in that demonstration and ★3·11 would refer to a proposition in the main body of the article; thus, $(1) . (2) .$ $★3{\cdot}11 . \supset .$ means that "the subsidiary propositions (1) and (2), together with the main proposition 3·11 imply" the protasis. The hypothesis of the proposition to be proved is cited by "Hp"; thus, Hp . (1) means "the hypothesis together with the subsidiary proposition (1)"; but Hp (1) means the hypothesis of the subsidiary proposition (1) and Hp (3·11) the hypothesis of the previous proposition ★3·11.

SECTION III.

On Finite and Infinite Cardinal Numbers.

This section III is entirely due to Russell, with the exception of some of the notes.

★1

·1 $$u \text{ sim } v . = . u, v \,\varepsilon\, \text{cls} . \exists\, 1 \rightarrow 1 \frown R \ni (u \supset \rho . v = \breve{\rho}u). \qquad \text{Df.}$$

Note: sim is contracted from similar, and u sim v can be read: u is similar to v. Thus sim is a symbol for a relation, and has all the properties ascribed to such symbols in Section II.

·2 $$u \,\varepsilon\, \text{cls} . \supset . \mu u = \text{cls} \frown v \ni (u \text{ sim } v). \qquad \text{Df.}$$

·3 $$Nc = \text{cls}^2 \frown z \ni \{\exists\, \text{cls} \frown u \ni (z = \mu u)\} \qquad \text{Df.}$$

·4 $$0 \;= \iota\Lambda. \qquad \text{Df.}$$

·41 $$1 \;= \text{cls} \frown u \ni (x \,\varepsilon\, u . \supset . u - \iota x \,\varepsilon\, 0). \qquad \text{Df.}$$

Note: 1 has the same signification as elm (cf. Section I) and will now replace that symbol.

★1·42 $2 = \text{cls} \cap u \ni (x \,\varepsilon\, u \,.\supset.\, u - \iota x \,\varepsilon\, 1)$. Df.

·43 $3 = \text{cls} \cap u \ni (x \,\varepsilon\, u \,.\supset.\, u - \iota x \,\varepsilon\, 2)$. Df.

·5 $n \,\varepsilon\, Nc \,.\supset.\, n + 1 - \text{cls} \cap u \,\varepsilon\, (x \,\varepsilon\, u \,.\supset.\, u - \iota x \,\varepsilon\, n)$. Df.

·6 $Nc \text{ fin} = Nc \cap n \ni \{s \,\varepsilon\, \text{cls} \,.\, 0 \,\varepsilon\, s : m \,\varepsilon\, Nc \cap s \,.\supset_m.\, m + 1 \,\varepsilon\, s :\supset.\, n \,\varepsilon\, s\}$. Df.

·61 $\text{Induct} \,.=.\!\cdot\!\cdot\, s \,\varepsilon\, \text{cls} \,.\, 0 \,\varepsilon\, s : m \,\varepsilon\, Nc \cap s \,.\supset_m.\, m + 1 \,\varepsilon\, s :\supset_s.\, Nc \text{ fin} \supset s$ Df.

Note: Induct is contracted from Induction.

·7 $0 \,\varepsilon\, Nc$ fin.

·71 $1 \,\varepsilon\, Nc$ fin, [$1 = 0 + 1 \,.\, 0 \,\varepsilon\, Nc \text{ fin} \,.\supset.\,$ Prop].

·72 $2 \,\varepsilon\, Nc$ fin.

·73 $n \,\varepsilon\, Nc \text{ fin} \,.\supset.\, n + 1 \,\varepsilon\, Nc$ fin, [Induct].

·8 $Nc \text{ infin} = Nc - Nc$ fin. Df.

1.81 cls fin $= \cup\, \text{‘}Nc$ fin. Df.

·82 cls infin $= \cup\, \text{‘}Nc$ infin. Df.

.83 cls $=$ cls fin $\cup$ cls infin.

·84 cls fin $\cap$ cls infin $= \Lambda$.

·91 $u \text{ sim } u' \,.\, v \text{ sim } v' \,.\, u \cap v = \Lambda \,.\, u' \cap v' = \Lambda \,.\supset.\, u \cup v \text{ sim } u' \cup v'$,
[$S,\, S' \,\varepsilon\, 1 \rightarrow 1 \,.\, u = \sigma \,.\, u' = \breve{\sigma} \,.\, v = \sigma' \,.\, v' = \breve{\sigma}' .\; S \cup S' = S'' \,.\supset.$
$S'' \,\varepsilon\, 1 \rightarrow 1 \,.\, \sigma'' \,\varepsilon\, u \cup v \,.\, \breve{\sigma}'' = u' \cup v' \,.\supset.\,$ Prop].

·92 $\alpha, \beta \,\varepsilon\, Nc \,.\supset.\, \alpha + \beta = \text{℩} Nc \cap z \ni \{u \,\varepsilon\, \alpha \,.\, v \,\varepsilon\, \beta \,.\, u \cap v = \Lambda \,.\supset.\, u \cup v \,\varepsilon\, z\}$. Df.

·93 $m \,\varepsilon\, Nc \,.\, n \,\varepsilon\, Nc \text{ fin} \,.\supset.\, m + (n + 1) = \text{℩} Nc \cap z \ni \{u \,\varepsilon\, m \,.\, v \,\varepsilon\, n \,.$
$u \cap v = \Lambda \,.\, x - \varepsilon\, u \cup v \,.\, v \cup \iota x = v' \,.\supset.\, u \cup v' \,\varepsilon\, z\}$.

·94 $(m + n) + 1 = \text{℩} Nc \cap z \ni \{w \,\varepsilon\, m + n \,.\, x - \varepsilon\, w \,.\supset.\, w \cup \iota x \,\varepsilon\, z\}$.

·95 $m + (n + 1) = (m + n) + 1$.

★2

·1 $n \,\varepsilon\, Nc \text{ fin} \,.\supset.\, n - = n + 1$,
[$0 - = 1 \;\ldots.\; (1) \quad n - = n + 1 \,.=:\, u \,\varepsilon\, n \,.\, x - \varepsilon\, u \,.\supset.\, u \cup \iota x - \text{sim } u :$
$=: u \cup \iota x \,\varepsilon\, n + 1 \,.\, y - \varepsilon\, u \cup \iota x \,.\supset.$
$u \cup \iota x \cup \iota y - \text{sim } u \cup \iota x :$
$=: n + 1 - = (n + 1) + 1$, (2)
(1) . (2) . Induct $.\supset.$ Prop].

·2 $R = \text{℩}\, 1 \rightarrow 1 \cap R' \ni (\rho' = Nc \text{ fin} : x R' y \,.=.\, x + 1 = y)$. Df.

Note: This special signification for R, and thus for ρ and $\breve{\rho}$, is only retained throughout Section III.

★2·3 $R \supset 0'$, [★2·1 . $\supset$. Prop] .

·31 $\breve{\rho} \supset \rho \,.\, \exists \rho - \breve{\rho}$, $[\breve{\rho} = \rho - \iota 0]$.

·32 $\mathrm{Induct} \,.=: s \,\varepsilon\, \mathrm{cls} \,.\, 0 \,\varepsilon\, s \,.\, \breve{\rho}(s \cap \rho) \supset s \,.\supset_s .\, \rho \supset s$.

·4 $\alpha_0 = \mathrm{cls} \cap s \ni \{\exists 1 \rightarrow 1 \cap S \ni (\breve{\sigma} \supset \sigma \,.\, \exists \sigma - \breve{\sigma} : s \supset \sigma \,.\, \breve{\sigma} s \supset s \,.\, \exists s - \breve{\sigma} s \,.\supset.\, s = \sigma)\}$. Df.

·41 $\alpha_0 \,\varepsilon\, Nc$, [cf. Russell, Révue de Math., loc. cit., §3, Prop. 1·9].

·42 $Nc \ \mathrm{fin} \,\varepsilon\, \alpha_0$, [★2·31 . ★2·32 . $\supset$. Prop] .

·43 $\alpha_0 \,\varepsilon\, Nc \ \mathrm{infin}$, $[\alpha_0 = \mu Nc \ \mathrm{fin} \,.\, Nc \ \mathrm{fin} = \rho \,.\, \mu\rho = \mu\breve{\rho} = \mu\breve{\rho} + 1$. ★2·1 . $\supset$. Prop] .

·5 $n \,\varepsilon\, Nc - \iota o \,.\supset.\, n - 1 = \imath\, Nc \cap x \ni (x + 1 = n)$. Df.

·51 $n \,\varepsilon\, Nc \ \mathrm{infin} \,.\supset.\, n - 1 \,\varepsilon\, Nc \ \mathrm{infin}$, [★1·8 . ★1·73 . $\supset$. Prop].

·6 $n \,\varepsilon\, Nc \ \mathrm{infin} \,.\, u \,\varepsilon\, n \,.\supset.\, \exists \alpha_0 \cap \mathrm{cls}'u$,

[$u \,\varepsilon\, n \,.\, x \,\varepsilon\, u$. ★2·51 . $\supset . \, u - \iota x \,\varepsilon\, \mathrm{cls\ infin}$, (1)

$m \,\varepsilon\, Nc \ \mathrm{fin} \,.\, u_m \,\varepsilon\, m \,.\, u_m \supset u \,.\, u - u_m \,\varepsilon\, \mathrm{cls\ infin} \,.\, x \,\varepsilon\, u - u_m \,.\supset.\, u - u_m - \iota x \,\varepsilon\, \mathrm{cls\ infin}$, (2)

(1) . (2) . Induct . $\supset : m \,\varepsilon\, Nc \ \mathrm{fin} \,.\, u_m \,\varepsilon\, m \,.\, u_m \supset u \,.\supset.\, u - u_m \,\varepsilon\, \mathrm{cls\ infin}$, (3)

$K = \mathrm{cls}'\mathrm{cls} \cap v \ni \{v \supset \mathrm{cls}'u \cap \mathrm{cls\ fin} :: m \,\varepsilon\, Nc \ \mathrm{fin} \,.\supset_m .\therefore v \cap m \,\varepsilon\, 1 : p, p' \,\varepsilon\, v \,.$

$p \,\varepsilon\, m \,.\, p' \,\varepsilon\, m + 1 \,.\supset.\, p \supset p'\} \,.\, v \,\varepsilon\, K \,.\supset.\, \cup{}'v \,\varepsilon\, \alpha_0 \,.\, \cup{}'v \supset u$. (4)

$(3) \supset \exists K \ldots.$ (5), (4) . (5) . $\supset$. Prop].

·7 $u \,\varepsilon\, \mathrm{cls\ infin} \,.\, v \,\varepsilon\, \mathrm{cls\ fin} \,.\, v \supset u \,.\supset.\, u \ \mathrm{sim}\ u - v$,

[★2·6 . $\supset . \, \exists \alpha_0 \cap \mathrm{cls}'u$, (1)

$w \,\varepsilon\, \alpha_0 \cap \mathrm{cls}'u \,.\, v' \,\varepsilon\, \mathrm{cls\ fin} \,.\, v' \supset w \,.\supset.\, w \ \mathrm{sim}\ w - v' \,.\supset.\, u \ \mathrm{sim}\ u - v'$, (2)

$v' \ \mathrm{sim}\ v \,.\supset.\, u - v' \ \mathrm{sim}\ u - v \ldots.$ (3), (2) . (3) . $\supset$. Prop].

·71 $\alpha \,\varepsilon\, Nc \ \mathrm{infin} \,.\, n \,\varepsilon\, Nc \ \mathrm{fin} \,.\supset.\, \alpha - n = \alpha = \alpha + n$, [★2·7 . $\supset$. Prop].

·72 $m, n \,\varepsilon\, Nc \ \mathrm{fin} \,.\supset.\, m + n - = m$, [★2·4 . $\supset$. Prop].

·73 $m + n \,\varepsilon\, Nc \ \mathrm{fin} \,.\supset.\, m, n \,\varepsilon\, Nc \ \mathrm{fin}$.

·74 $m, n \,\varepsilon\, Nc \,.\supset: m + n = m \,.=.\, m \,\varepsilon\, Nc \ \mathrm{infin}$.

·75 $Nc \ \mathrm{fin} = Nc \cap n \ni (n - = n + 1)$, [★2·1 . ★2·71 . Transp . $\supset$. Prop].

·76 $Nc \ \mathrm{infin} = Nc \cap \alpha \ni (\alpha = \alpha - 1)$, [★2·1 . Transp . ★2·71 . $\supset$. Prop].

The Props ★2·75 and ★2·76 show that from the Df of finite numbers by induction we can deduce the properties of infinity with regard to similarity of whole and part.

★3

·1 $\alpha, \beta \,\varepsilon\, Nc \,.\supset: \alpha < \beta \,.=.\, \alpha \cap \beta = \Lambda \,.\, \exists \alpha \cap u \ni \{\exists \beta \cap v \ni (u \supset v)\}$. Df.

·12 $\alpha, \beta \,\varepsilon\, Nc \,.\supset: \alpha > \beta \,.=.\, \beta < \alpha$, Df.

·12 $\alpha, \beta \,\varepsilon\, Nc \,.\supset: \alpha \leqq \beta \,.=.\, \exists \alpha \cap u \ni \{\exists \beta \cap v \ni (u \supset v)\}$. Df.

★3·2 $\alpha, \beta \varepsilon Nc . \alpha < \beta . v \varepsilon \beta . \supset . \exists \alpha \cap \text{cls'}v$

$[u_0 \varepsilon \alpha . v_0 \varepsilon \beta . u_0 \supset v_0 . v \varepsilon \beta . \supset . \exists 1 \rightarrow 1 \cap S \ni (\sigma = v_0 . \breve{\sigma} = v . \breve{\sigma} u_0 \varepsilon \alpha \cap \text{cls'}v)]$.

·3 $m, n \varepsilon Nc \text{ fin} . \supset : m \leqq n . = . m R^{n-m} n,$ [cf. ★2·2],

$[a, b \varepsilon Nc \text{ fin} . aRb . \supset . \exists a \cap u \ni (x - \varepsilon u . \supset . u \cup \iota x \varepsilon b) . \supset : v \varepsilon b .$

$\supset_v . \exists a \cap \text{cls'}v,$ (1)

$a, b, c \varepsilon Nc \text{ fin} . aR^c b . v \varepsilon b . \exists a \cap \text{cls'}v . \supset : x - \varepsilon v . \supset . \exists a \cap \text{cls'}(v \cup \iota x).$ (2)

$(1) . (2) . \text{Induct} . \supset : a, b, c \varepsilon Nc \text{ fin} . aR^c b . v \varepsilon b . \supset . \exists a \cap \text{cls'}v,$ (3)

$aRb . \supset . a = b - 1 . 1 = b - a.$ (4)

$aR^c b . \supset . c = b - a : \supset : aR^{c'}(b + 1) . \supset . c' = b + 1 - a,$ (5)

$(4) . (5) . \text{Induct} . \supset : aR^c b . \supset . c = b - a,$ (6)

$(3) . (6) . \supset : a, b, c \varepsilon Nc \text{ fin} . aR^{b-a} b . \supset . a \leqq b,$ (7)

$a = b . \supset . aR^0 b . \supset . aR^{b-a} b,$ (8)

$a < b . u \varepsilon a . v \varepsilon b . u \supset v . v - u \varepsilon c . \supset . aR^c b . c = b - a,$ (9)

$(7) . (8) . (9) . \supset . \text{Prop}]$.

Note: This prop. connects the cardinal and ordinal theories of finite number.

★4

·1 $\alpha \varepsilon Nc \text{ infin} . \supset . \alpha \geqq \alpha_0 .$ [★2·6 . $\supset$. Prop].

·2 $\alpha_0 = \alpha_0 + \alpha_0,$

$[R' = \text{ɿRel} \cap P \ni \{xPy . = . \exists Nc \text{ fin} \cap n \ni (xR^{2n}y)\} .$

$\rho_0 = \breve{\rho}'0 . \rho_1 = \breve{\rho}'1 . \supset . \rho_0 \cap \rho_1 = \Lambda . \rho_0 \cup \rho_1 = \rho . \rho_0, \rho_1 \varepsilon \alpha_0 . \supset . \text{Prop}]$.

·21 $n \varepsilon Nc \text{ fin} . \supset . \alpha_0 = \alpha_0 \times n,$ [★4·2 . Induct . $\supset$. Prop].

·22 $\alpha_0 = \alpha_0 \times \alpha_0,$

$[M = x \ni \{\exists Nc \text{ fin} \cap (m, n) \ni (x = \iota m \cup \iota n)\} . \supset . \mu M = \alpha_0 \times \alpha_0,$ (1)

$R' \varepsilon 1 \rightarrow 1 . \rho' = M .\cdot. \iota m \cup \iota n R' \iota m' \cup \iota n' . = : m > n . m - n > 1 .$

$m' + n' = m + n . m' = m - 1 . \cup . m \geqq n . m - n \leqq 1 .$

$m' + n' = m + n + 1 . n' = 0 .\cdot. \supset . \rho' \varepsilon \alpha_0 . \supset . \text{Prop}]$.

·23 $n \varepsilon Nc \text{ fin} . \supset . \alpha_0 = \alpha_0^n,$ [Induct].

·24 $u, v \varepsilon \text{cls'}Nc \text{ fin} . \supset . \sum^{x \epsilon u . y \epsilon v} x \times \alpha^y = \alpha_0 .$

·3 ($u \varepsilon$ cls infin $. \supset . \exists$ cls'α_0 excl $\cap v \ni (\cup{}^{\backprime}v = u)$. Pp.

For definition of cls'α_0 excl, cf ★5·1 later.)

Note: It has been shown that every infinite class contains classes whose number is α_0. Thus if a class v of α_0's has been found which contains all but a finite number of u's, these may be added on to the v's; but if an infinite number of u's remain, new α_0's may be found out of them, and to this process there can be no end as long as any u's remain. It appears not impossible that a theorem may be discovered proving that this process can always be completed. In any case, it is much to be desired that the possibility in all cases of this completion of the above process should be either proved or disproved. In order to show the importance of this investigation, the theorem is here hypothetically assumed and stated as a primitive proposition (★4·3), and some immediate deductions (★4·31 to ★4·51) are made from it. The subject is resumed in the sets of propositions ★30, ★31, and ★32.

$\alpha \,\varepsilon\, Nc$ infin . ★4·3 . $\supset$:

★4·31 $\exists\, Nc \cap \beta \ni (\alpha = \alpha_0 \beta)$, [★4·3 . $\supset$. Prop].

·32 $\alpha + \alpha = \alpha$, $[\alpha = \alpha_0\beta = (\alpha_0 + \alpha_0)\,\beta = \alpha + \alpha]$.

·33 $\alpha\alpha_0 = \alpha$, $[\alpha = \alpha_0\beta = \alpha_0^2\beta = \alpha\alpha_0]$.

·34 $\alpha \,\varepsilon\, Nc \cap \beta \ni (\alpha = \alpha_0\beta)$, [★4·33 . $\supset$. Prop].

·35 $\alpha > \alpha_0 \,.\, \beta < \alpha \,.\, \supset .\, \alpha_0\beta < \alpha$,

[$\beta \,\varepsilon\, Nc$ fin . $\supset .\, \alpha_0\beta = \alpha_0 \,.\, \supset .\, \alpha_0\beta < \alpha$, (1)

$\beta \,\varepsilon\, Nc$ infin . $\supset .\, \alpha_0\beta = \beta \,.\, \supset .\, \alpha_0\beta < \alpha$, (2)

(1) . (2) . $\supset$. Prop].

·36 $\alpha > \alpha_0 \,.\, \supset .\, Nc \cap \beta \ni (\alpha = \alpha_0\beta) = \iota\alpha$, [★4·35 . $\supset$. Prop].

·37 $\beta < \gamma \,.\, \supset .\, \alpha + \beta \leqq \alpha + \gamma$,

$[u \,\varepsilon\, \alpha \,.\, v \,\varepsilon\, \beta \,.\, w \,\varepsilon\, \gamma \,.\, v \supset w \,.\, u \cap w = \Lambda \,.\, \supset .\, u \cup v \supset u \cup w \,.\, \supset .$ Prop].

·38 $\beta < \alpha \,.\, \supset .\, \alpha + \beta = \alpha$, $[\alpha \leqq \alpha + \beta \leqq \alpha + \alpha \,.\, \alpha = \alpha + \alpha \,.\, \supset .$ Prop].

·39 $\beta < \gamma \,.\, \supset .\, \alpha\beta \leqq \alpha\gamma$,

$[v \,\varepsilon\, \beta \,.\, w \,\varepsilon\, \gamma \,.\, v \supset w \,.\, w \supset \alpha \,.\, \supset .\, \cup\,'v \supset \cup\,'w \,.\, \supset .$ Prop].

·4 $\beta < \gamma \,.\, \supset .\, \alpha^\beta \leqq \alpha^\gamma$,

$[u \,\varepsilon\, \alpha \,.\, v \,\varepsilon\, \beta \,.\, w \,\varepsilon\, \gamma \,.\, v \supset w \,.\, \supset .\, \alpha^\beta = \mu Nc \to 1 \cap S \ni (\sigma = v \,.\, \breve{\sigma} \supset u) \,.\, \supset .$

$\alpha^\gamma = \mu Nc \to 1 \cap S \ni (\sigma = w \,.\, \breve{\sigma} \supset u) \,.\, \supset .$ Prop].

Note: Here Cantor's definition of the power of a cardinal number (e. g. α^β) is used, cf. ★14·1 of Section V.

·41 $\beta < \gamma \,.\, \supset .\, \beta^\alpha \leqq \gamma^\alpha$,

$[u \,\varepsilon\, \alpha \,.\, v \,\varepsilon\, \beta \,.\, w \,\varepsilon\, \gamma \,.\, v \supset w \,.\, \supset .\, Nc \to 1 \cap S \ni (\sigma = u \,.\, \breve{\sigma} \supset v) \,\varepsilon\, \beta^\alpha \,.$

$Nc \to 1 \cap S \ni (\sigma = u \,.\, \breve{\sigma} \supset w) \,\varepsilon\, \gamma^\alpha$, (1)

(1) . $Nc \to 1 \cap S \ni (\sigma = u \,.\, \breve{\sigma} \supset v) \supset Nc \to 1 \cap S \ni (\sigma = u \,.\, \breve{\sigma} \supset w) \,.\, \supset .$ Prop].

·42 $n \,\varepsilon\, Nc$ fin . $\supset .\, n\alpha = \alpha \,.\, \alpha\alpha_0^n = \alpha$.

·43 $\beta, \gamma \,\varepsilon\, Nc \,.\, \beta \leqq \alpha \,.\, \supset .\, \alpha^\gamma = (\alpha + \beta)^\gamma$.

·44 $\beta \,\varepsilon\, Nc \,.\, \gamma \,\varepsilon\, Nc$ infin . $\beta \leqq \alpha \,.\, \supset .\, \alpha^\gamma\beta^\gamma = \alpha^\gamma$.

[This theorem is enunciated and proved in Section V, ★20·1; it is mentioned here for the sake of completeness.]

★4·5 $u, v \varepsilon \text{cls} \cdot \exists \mu u \frown \text{cls'} v \cdot \exists \mu v \frown \text{cls'} u \cdot \supset \cdot u \text{ sim } v.$

Note: This is Schröder's and Bernstein's theorem; cf. Borel, "Théorie des Functions," Paris, 1898, p. 108 ff.

$[S \varepsilon 1 \rightarrow 1 \cdot \sigma = u \cdot \breve{\sigma} \supset v \cdot S' \varepsilon 1 \rightarrow 1 \cdot \sigma' = v \cdot \breve{\sigma}' \supset u \cdot \supset \cdot \sigma(\sigma') = u \cdot \breve{\sigma}'(\breve{\sigma}) \supset \breve{\sigma}' \supset u,$ (1)

$(1) \cdot \mu u = \alpha \cdot \mu v = \beta \cdot \mu u - \breve{\sigma}' = \gamma \cdot \mu \breve{\sigma}' - \breve{\sigma}'(\breve{\sigma}) = \delta \cdot \supset \cdot$

$\alpha \geqq \beta \cdot \beta \geqq \alpha \cdot \alpha = \beta + \gamma \cdot \beta = \alpha + \delta \cdot \supset \cdot \alpha = \alpha + \gamma + \delta,$ (2)

$(2) \cdot$ ★$4{\cdot}38 \cdot \supset \cdot \gamma + \delta \leqq \alpha \cdot \supset \cdot \delta \leqq \alpha \cdot \supset \cdot \alpha + \delta = \alpha \cdot \supset \cdot \beta = \alpha \cdot \supset \cdot$ Prop].

Note: Bernstein's proof of Schröder's and Bernstein's theorem given by Borel (loc. cit.) does not depend upon ★4·3.

·51 $\alpha, \beta \varepsilon Nc \cdot \alpha > \beta \cdot \supset \cdot -(\beta \geqq \alpha),$ [★$4{\cdot}5 \cdot \supset \cdot$ Prop].

SECTION IV.

★5

·1 $\text{cls}^2 \text{ excl} \cdot = \cdot \text{cls}^2 \frown d \ni \{q, q' \varepsilon d \cdot \supset \cdot q = q' \cdot \smile \cdot q \frown q' = \Lambda\}.$ Df.

Note: excl is contracted from "exclusive".

·2 $a \varepsilon \text{cls}^2 \cdot \supset \cdot \text{cls'} a \text{ excl} = \text{cls'} a \frown \text{cls}^2 \text{ excl}.$ Df.

★6 $d \varepsilon \text{cls}^2 \text{ excl} \cdot \supset \therefore$

·0 $d^{\times} = \text{cls} \frown m \ni \{p \varepsilon d \cdot \supset_p \cdot p \frown m \varepsilon 1 : m \supset \smile \text{‘}d\}.$ Df.

Note: $d^{\times}$ is the "multiplicative" class of d, and can be read as "d multiplicative": it has no existence, according to this definition, unless d is a cls² excl.

·11 $p \varepsilon d \cdot x \varepsilon p \cdot d_1 = d - \iota p \cdot m_1 \varepsilon d_1^{\times} \cdot m = m_1 \smile \iota x \cdot \supset \cdot m \varepsilon d^{\times},$

$[\text{Hp} \cdot p' \varepsilon d \cdot p' -= p \cdot \supset \cdot p' \varepsilon d_1 \cdot x - \varepsilon p' \cdot \supset \cdot m \frown p' = m_1 \frown p' \cdot \supset \cdot m \frown p' \varepsilon 1.$ (1)

$\text{Hp} \cdot \supset \cdot m \frown p = \iota x \cdot \supset \cdot m \frown p \varepsilon 1.$ (2)

$(1) \cdot (2) \cdot \supset : p'' \varepsilon d \cdot \supset \cdot p'' \frown m \varepsilon 1 \ldots . (3)$ $m \supset \smile \text{‘}d,$ (4)

$(3) \cdot (4) \cdot \supset \cdot$ Prop].

·12 $u \supset \smile \text{‘}d : x, y \varepsilon u \cdot x 0' y \cdot \supset_{x,y} \cdot - \exists d \frown p \ni (x, y \varepsilon p) : \supset \cdot \exists d^{\times} \frown m \ni (u \supset m),$

$[\text{Hp} \cdot p \varepsilon d \cdot \exists p \frown u \cdot \supset \cdot p \frown u \varepsilon 1,$ (1)

$d_u = d \frown p \ni (p \frown u = \Lambda) \cdot \supset \cdot d_u \varepsilon \text{cls}^2 \text{ excl} \cdot \supset \cdot \exists d_u^{\times},$ (2)

$(1) \cdot m_1 \varepsilon d_u^{\times} \cdot m = m_1 \smile u \cdot \supset : p \varepsilon d \cdot \supset_p \cdot p \frown m \varepsilon 1,$ (3)

$m \supset \smile \text{‘}d \ldots . (4),$ $(3) \cdot (4) \cdot \supset \cdot m \varepsilon d^{\times} \cdot \supset \cdot$ Prop].

·13 $m \varepsilon d^{\times} \cdot \supset \cdot - \exists d^{\times} \frown m' \ni (m' o' m \cdot m' \supset m),$

$[m' o' m \cdot m' \supset m \cdot \supset \cdot \exists d \frown p \ni (m' \frown p = \Lambda) \cdot \supset \cdot m' - \varepsilon d^{\times} \cdot \supset \cdot$ Prop].

$\bigstar 6{\cdot}14 \quad l \,\varepsilon\, \mathrm{cls} \,.\supset .\!\cdot\!.\, d^{\times} \cap \mathrm{cls}\text{'}l \,\varepsilon\, 1 \,.=:\, p \,\varepsilon\, d \,.\supset_p .\, p \cap l \,\varepsilon\, 1,$

$[\bigstar 6{\cdot}0 \,.\, \mathrm{Hp} \,.\supset:\, p \,\varepsilon\, d \,.\supset_p .\, \exists p \cap l,$ (1)

$m = \text{ɿ}d^{\times} \cap \mathrm{cls}\text{'}l \,.\, p \,\varepsilon\, d \,.\, x = \text{ɿ}p \cap m \,.\supset .\, x \,\varepsilon\, l,$ (2)

$\mathrm{Hp}(2) \,.\, y \,\varepsilon\, p \cap l \,.\supset .\, m - \iota x \cup \iota y \,\varepsilon\, d^{\times} \cap \mathrm{cls}\text{'}l \,.\supset .\, m - \iota x \cup \iota y = m \,.\supset .\, x1\text{'}y,$ (3)

$(1) \,.\, (2) \,.\, (3) \,.\supset .\!\cdot\!.\, d^{\times} \cap \mathrm{cls}\text{'}l \,\varepsilon\, 1 \,.\supset:\, p \,\varepsilon\, d \,.\supset_p .\, p \cap 1 \,\varepsilon\, 1$ (4)

$\bigstar 6{\cdot}0 \,.\supset .\!\cdot\!.\, p \,\varepsilon\, d \,.\supset_p .\, p \cap l \,\varepsilon\, 1 :\supset .\, d^{\times} \cap \mathrm{cls}\text{'}l \,\varepsilon\, 1,$ (5)

$(4) \,.\, (5) \,.\supset .\, \mathrm{Prop}].$

$\cdot 21 \quad \cup\,\text{‘}d^{\times} = \cup\,\text{‘}d.$

$\cdot 22 \quad f \,\varepsilon\, \mathrm{cls}^2\,\mathrm{excl} \,.\supset:\, d^{\times} = f^{\times} \,.=.\, d = f,$

$[\bigstar 6{\cdot}21 \,.\, \mathrm{Hp} \,.\supset .\, \cup\,\text{‘}d = \cup\,\text{‘}f,$ (1)

$p \,\varepsilon\, d \,.\, x, y \,\varepsilon\, p \,.\supset .\, -\exists d^{\times} \cap m \ni (x, y \,\varepsilon\, m) \,.\supset .\, -\exists f^{\times} \cap m \ni (x, y \,\varepsilon\, m),$ (2)

$x, y \,\varepsilon^2 f \,.\, -\exists f^{\times} \cap m \ni (x, y \,\varepsilon\, m) \,.\supset .\, \exists f \cap q \ni (x, y \,\varepsilon\, q).$ (3)

$(1) \,.\, (2) \,.\, (3) \,.\supset:\, d^{\times} = f^{\times} \,.\supset .\, d = f,$ (4)

$(4) :\, d = f \,.\supset .\, d^{\times} = f^{\times} :\supset .\, \mathrm{Prop}].$

$\bigstar 7$

$\cdot 0 \quad R \,\varepsilon\, 1 \rightarrow 1 \,.\, R \supset \mathrm{sim} \,.\, \rho, \breve{\rho} \,\varepsilon\, \mathrm{cls}^2\,\mathrm{excl} \,.\supset .\, \cup\,\text{‘}\rho \;\mathrm{sim}\; \cup\,\text{‘}\breve{\rho},$

$[u \,\varepsilon\, \rho \,.\supset .\, \exists 1 \rightarrow 1 \cap R' \ni (\rho' = u \,.\, \breve{\rho}' = \text{ɿ}\breve{\rho}u),$ (1)

$K = x \ni [\exists \rho \cap u \ni \{x = 1 \rightarrow 1 \cap R' \ni (\rho' = u \,.\, \breve{\rho}' = \text{ɿ}\breve{\rho}u)\}].$

$L \,\varepsilon\, K^{\times} \,.\, S = \cup\,\text{‘}L \,.\supset .\, S \,\varepsilon\, 1 \rightarrow 1 \,.\, \sigma = \cup\,\text{‘}\rho \,.\, \breve{\sigma} = \cup\,\text{‘}\breve{\rho} \,.\supset .\, \mathrm{Prop}].$

This prop proves that if d is a cls^2 excl, $\mu \cup \text{‘}d$ is defined entirely by the numbers involved in d and its classes; thus we get the conception of a sum of numbers.

$\cdot 1$
$$d \,\varepsilon\, \mathrm{cls}^2\,\mathrm{excl} \,.\supset .\, \overset{p \,\varepsilon\, d}{\sum} \mu p = \mu \cup \text{‘}d. \qquad \text{Df.}$$

Thus $\overset{p \,\varepsilon\, d}{\sum} \mu p$ is the sum of all the numbers of the classes forming the extension of the class d. The usual notation $\alpha + \beta + \gamma + \ldots.$ is only suitable for a finite, or at least for a well-ordered set of numbers; also, it introduces unnecessarily the idea of order among the numbers added, and thus renders necessary the totally irrelevant commutative law. But we shall use it, when convenient (e. g. Section III above), for a finite number of numbers; thus,

$$\alpha, \beta \,\varepsilon\, Nc \,.\, p \,\varepsilon\, \alpha \,.\, q \,\varepsilon\, \beta \,.\, d = \iota p \cup \iota q \,.\supset .\, \overset{u \,\varepsilon\, d}{\sum} \mu u = \alpha + \beta = \beta + \alpha = \mu \cup \text{‘}d.$$

$\cdot 2 \quad R \,\varepsilon\, 1 \rightarrow 1 \,.\, R \supset \mathrm{sim} \,.\, \rho, \breve{\rho} \,\varepsilon\, \mathrm{cls}^2\,\mathrm{excl} \,.\supset .\, \rho^{\times} \;\mathrm{sim}\; \breve{\rho}^{\times},$

$[u \,\varepsilon\, \rho \,.\supset .\, \exists 1 \rightarrow 1 \cap R' \ni (\rho' = u \,.\, \breve{\rho}' = \text{ɿ}\breve{\rho}u).$ (1)

$K = x \ni [\exists \rho \cap u \ni \{x = 1 \rightarrow 1 \cap R' \ni (\rho' = u \,.\, \breve{\rho}' = \text{ɿ}\breve{\rho}u)\}] \,.\, L \,\varepsilon\, K^{\times}.$

$S = \cup\,\text{‘}L \,.\supset .\!\cdot\!.\, m \,\varepsilon\, \rho^{\times} \,.\supset_m .\, \breve{\rho}^{\times} \cap n \ni (n = \breve{\sigma}m) \,\varepsilon\, 1 :\, n \,\varepsilon\, \breve{\rho}^{\times} \,.\supset_n .$

$\rho^{\times} \cap m \ni (m = \sigma n) \,\varepsilon\, 1 .\!\cdot\!. \supset .\, \mathrm{Prop}].$

Analogous remarks to those on ★7·0 apply here; we derive from this prop the conception of a product of numbers.

★7·21 $d \,\varepsilon\, \text{cls}^2\, \text{excl} \,.\supset.\, \prod^{p \varepsilon d} \mu p = \mu d^{\times}.$ Df.

Thus, $\prod^{p \varepsilon d} \mu p$ is the product of all the numbers of the classes forming the extension of the class d. Analogous remarks to those on ★7·1 apply to the notation $\alpha \times \beta \times \gamma$ .... for a product of cardinal numbers.

·3 $\alpha, \beta \,\varepsilon\, Nc \,.\, d \,\varepsilon\, \alpha \frown \text{cls}'\beta\ \text{excl} \,.\supset.\, \smile {}^{\backprime}d \,\varepsilon\, \alpha \times \beta,$

$[d \supset \beta \,.\supset:\, p, p' \varepsilon d \,.\supset.\, \exists 1 \rightarrow 1 \frown R \ni (p' = \rho \,.\, p = \breve{\rho}),$ (1)

$p \,\varepsilon\, d \,.\supset.\, R(p) = x \ni \{\exists d \frown p' \ni [x = 1 \rightarrow 1 \frown R \ni (p' = \rho \,.\, p = \breve{\rho})]\}:$

$P \,\varepsilon\, R(p)^{\times} .\, S = \smile {}^{\backprime}P \,.\, k = x \ni \{\exists p \frown y \ni (x = \sigma y)\} :\supset.\!\cdot\!.\, \sigma = \smile {}^{\backprime}d \,.$

$\breve{\sigma} = p \,.\, k \varepsilon \beta \frown \text{cls}'\alpha\ \text{excl} \,.\, k \supset d^{\times},$ (3)

$(3) \,.\, h = \iota d \smile \iota k \,.\supset.\, h^{\times} \,\varepsilon\, \alpha \times \beta,$ (4)

$k \supset d^{\times} .\, x \,\varepsilon\, h^{\times} .\supset.\, \exists d \frown p' \ni \{\exists k \frown q \ni (x = \iota p' \smile \iota q \,.$

$p' \frown q \,\varepsilon\, 1 \,.\, {\sim} p' \frown q \,\varepsilon \smile {}^{\backprime}d)\},$ (5)

$y \,\varepsilon \smile {}^{\backprime}d \,.\supset.\, h^{\times} \frown x \ni (\frown {}^{\backprime}x = \iota y) \,\varepsilon\, 1,$ (6)

$(4) \,.\, (5) \,.\, (6) \,.\supset.$ Prop].

Note: The proposition connects addition with the multiplication of two numbers. It also gives a signification to the commutative law $\alpha \times \beta = \beta \times \alpha$, namely, it can be construed to signify

$$d \,\varepsilon\, \alpha \frown \text{cls}'\beta\ \text{excl} \,.\, d' \,\varepsilon\, \beta \frown \text{cls}'\alpha\ \text{excl} \,.\supset.\, \smile {}^{\backprime}d \text{ sim } \smile {}^{\backprime}d'.$$

Also incidentally the demonstration shows how to construct a cls^2 excl contained in $d^{\times}$; for $k \,\varepsilon\, \text{cls}^2$ excl and $k \supset d^{\times}$.

★8 $((a)) \,\varepsilon\, \text{cls}^3 .\supset.\!\cdot\!.$

·1 $(\smile {}^{\backprime}(a)) = q \ni \{\exists\, ((a)) \frown p \ni (q = \smile {}^{\backprime}p)\}.$ Df.

Note: In $((a))$ the symbol a is devoid of independent signification.

·11 $\smile {}^{\backprime}(\smile {}^{\backprime}(a)) = \smile {}^{\backprime} \smile {}^{\backprime}((a)).$

·12 $p, q \,\varepsilon\, ((a)) \,.\, p \mathbin{-\!=} q \,.\supset_{p,q}.\, \smile {}^{\backprime}p \mathbin{-\!=} \smile {}^{\backprime}q :\supset.\, \mu(\smile {}^{\backprime}(a)) = \mu((a)).$

$[\text{Hp} \,.\, x \,\varepsilon\, (\smile {}^{\backprime}(a)) \,.\supset.\, ((a)) \frown y \ni (x = \smile {}^{\backprime}y) \,\varepsilon\, 1,$ (1)

$\text{Hp} \,.\, y \,\varepsilon\, ((a)) \,.\supset.\, (\smile {}^{\backprime}(a)) \frown x \ni (x = \smile {}^{\backprime}y) \,\varepsilon\, 1,$ (2)

$(1) \,.\, (2) \,.\supset.$ Prop].

·2 $\text{cls}^3\, \text{excl} = \text{cls}^3 \frown \text{cls}^2\, \text{excl}.$ Df.

·21 $\text{cls}^3\, \text{excl excl} = \text{cls}^3\, \text{excl} \frown \text{cls}'\,(\text{cls}^2\, \text{excl}).$ Df.

·22 $((a)) \,\varepsilon\, \text{cls}^3\, \text{arithm} \,.=.\, (\smile {}^{\backprime}(a)) \,\varepsilon\, \text{cls}^2\, \text{excl} \,.\, ((a)) \supset \text{cls}^2\, \text{excl}.$ Df.

Note: "Arithm" is contracted from "arithmetical"; this definition is essential to the statement and proof of the associative laws in their full generality.

★8·23 $((a))\,\varepsilon\,\mathrm{cls}^3\,\mathrm{arithm}\,.\supset.\cup\text{‘}((a))\,\varepsilon\,\mathrm{cls}^2\,\mathrm{excl},$

$[\bigstar 8{\cdot}22\,.\,\mathrm{Hp}\,.\supset:p\,\varepsilon\,((a))\,.\,q,q'\,\varepsilon\,p\,.\,q -= q'\,.\supset.\,q\cap q'=\Lambda,$ (1)

$\bigstar 8{\cdot}22\,.\,\mathrm{Hp}\,.\,p,p'\,\varepsilon\,((a))\,.\,p -= p'\,.\supset.\cup\text{‘}p\cap\cup\text{‘}p'=\Lambda\,.\supset:$

$q\,\varepsilon\,p\,.\,q'\,\varepsilon\,p'\,.\supset_{q,q'}.\,q\cap q'=\Lambda,$ (2)

$(1)\,.\,(2)\,.\supset.\,\mathrm{Prop}].$

·3 $((a))\,\varepsilon\,\mathrm{cls}^3\,\mathrm{arithm}\,.\supset.\sum^{p\,\epsilon\,((a))}\sum^{q\,\epsilon\,p}\mu q=\mu\cup\text{‘}(\cup\text{‘}(a)).$ Df.

·31 $((a))\,\varepsilon\,\mathrm{cls}^3\,\mathrm{arithm}\,.\supset.\sum^{p\,\epsilon\,((a))}\sum^{q\,\epsilon\,p}\mu q=\sum^{q\,\epsilon\,\cup\text{‘}((a))}\mu q,$ $[\bigstar 8{\cdot}11\,.\supset.\,\mathrm{Prop}].$

Note: This is the associative law for addition when the number of brackets and the number of numbers in each bracket is not necessarily finite or denumerable.

★9 $((a))\,\varepsilon\,\mathrm{cls}^3\,.\,((a))\supset\mathrm{cls}^2\,\mathrm{excl}\,.\supset.\because$

·0 $((a)^{\times})=m\ni\{\exists\,((a))\cap p\ni(m=p^{\times})\}.$ Df.

·1 $((a)^{\times})=((c))\,.\supset.\,(\cup\text{‘}(c))=(\cup\text{‘}(a)),$ $[\bigstar 6{\cdot}21\,.\supset\mathrm{Prop}].$

★10 $((a))\,\varepsilon\,\mathrm{cls}^3\,\mathrm{arithm}\,.\supset.\because$

·0 $\mu\,((a)^{\times})=\mu\,((a))=\mu\,(\cup\text{‘}(a)),$

$[\mathrm{Hp}\,.\,\bigstar 8{\cdot}12\,.\supset.\,\mu\,((a))=\mu\,(\cup\text{‘}(a)),$ (1)

$\mathrm{Hp}\,.\,p,q\,\varepsilon\,((a)^{\times})\,.\,p -= q\,.\supset.\cup\text{‘}p -= \cup\text{‘}q,$ (2)

$(2)\,.\,\bigstar 8{\cdot}12\,.\,\bigstar 9{\cdot}1\,.\,((a)^{\times})=((c))\,.\supset.\,\mu\,((a)^{\times})=\mu\,(\cup\text{‘}(c))=\mu\,(\cup\text{‘}(a));$ (3)

$(1)\,.\,(3)\,.\supset.\,\mathrm{Prop}].$

·1 $((a)^{\times})\,\varepsilon\,\mathrm{cls}^3\,\mathrm{excl}.$

·11 $p,q\,\varepsilon\,((a)^{\times})^{\times}\,.\,p -= q\,.\supset.\cup\text{‘}p -= \cup\text{‘}q,$

$[m,m'\,((a)^{\times})\,.\supset.\cup\text{‘}m\cap\cup\text{‘}m'=\Lambda,$ (1)

$\mathrm{Hp}\,.\supset.\,\exists\,((a)^{\times})\cap m\ni(p\cap m -= q\cap m),$ (2)

$\mathrm{Hp}\,.\,m\,\varepsilon\,((a)^{\times})\,.\,p\cap m -= q\cap m\,.\,u=\iota p\cap m\,.$

$v=\iota q\cap m\,.\supset:\exists u-v\,.\cup.\,\exists v-u,$ (3)

$(1)\,.\,\mathrm{Hp}\,(1)\,.\,\mathrm{Hp}\,(3)\,.\supset.\,u\cap\iota(q\cap m')=\Lambda\,.\,v\cap\iota(p\cap m')=\Lambda,$ (4)

$(2)\,.\,(3)\,.\,(4)\,.\supset:\exists\cup\text{‘}p-\cup\text{‘}q\,.\cup.\,\exists\cup\text{‘}q-\cup\text{‘}p:\supset.\,\mathrm{Prop}].$

·12 $((a)^{\times})^{\times}=((b))\,.\supset.\,\mu\,(\cup\text{‘}(b))=\mu\,((a)^{\times})^{\times},$

$[\bigstar 8{\cdot}12\,.\,\bigstar 10{\cdot}11\,.\supset.\,\mathrm{Prop}].$

★ 10·13 $\smile\text{‘}\smile\text{‘}((a)^{\times}) = \smile\text{‘}\smile\text{‘}((a)),$

$[y \,\varepsilon \smile\text{‘}((a)^{\times}) \,.=.\, \exists\, ((a)) \frown p \ni (y \,\varepsilon\, p^{\times}),$ (1)

$z \,\varepsilon \smile\text{‘}\smile\text{‘}((a)^{\times}) \,.=.\, \exists\, ((a)) \frown p \ni (z \,\varepsilon \smile\text{‘}p^{\times}),$ (2)

$(2) \,.\, ★ 6{\cdot}21 \,.\supset:\, z \,\varepsilon \smile\text{‘}\smile\text{‘}((a)^{\times}) \,.=.\, \exists\, ((a)) \frown p \ni (z \,\varepsilon \smile\text{‘}p) :\supset .\, \text{Prop}]$

·131 $q,\, q' \,\varepsilon \smile\text{‘}((a)^{\times}) \,.\supset:\, q -= q' \,.\supset.\, \smile\text{‘}q -= \smile\text{‘}q',$

$[p_1,\, p_2 \,\varepsilon\, ((a)) \,.\, p_1 -= p_2 \,.\, q \,\varepsilon\, p_1^{\times} \,.\, q' \,\varepsilon\, p_2^{\times} \,.\supset.\, \smile\text{‘}q -= \smile\text{‘}q'$ (1)

$p \,\varepsilon\, ((a)) \,.\, q,\, q' \,\varepsilon\, p^{\times} \,.\, p \,\varepsilon\, \text{cls}^2\,\text{excl} \,.\supset.\, \smile\text{‘}q -= \smile\text{‘}q',$ (2)

$(1) \,.\, (2) \,.\supset.\, \text{Prop}].$

·14 $((a)^{\times})^{\times} = ((b)) \,.\supset.\, (\smile\text{‘}(b)) = \{\smile\text{‘}((a))\}^{\times},$

$[★ 6{\cdot}0 \,.\, ★ 10{\cdot}11 \,.\supset:\, x \,\varepsilon\, (\smile\text{‘}(b)) \,.=.\, ((b)) \frown y \ni (x = \smile\text{‘}y) \,\varepsilon\, 1 \,.$

$=.\, y \ni \{y \supset \smile\text{‘}((a)^{\times}) : p \,\varepsilon\, ((a)^{\times}) \,.\supset_p.\, y \frown p \,\varepsilon\, 1 : x = \smile\text{‘}y\} \,\varepsilon\, 1.$ (1)

$★ 6{\cdot}0 \,.\, ★ 10{\cdot}13 \,.\, ★ 10{\cdot}131 \,.\supset:\, x \,\varepsilon\, \{\smile\text{‘}((a))\}^{\times} \,.=.$

$x \supset \smile\text{‘}\smile\text{‘}((a)) : q \,\varepsilon \smile\text{‘}((a)) \,.\supset_q.\, x \frown q \,\varepsilon\, 1 :=:$

$x \supset \smile\text{‘}\smile\text{‘}((a)^{\times}) : q \,\varepsilon \smile\text{‘}((a)) \,.\supset_q.\, x \frown q \,\varepsilon\, 1 :=:$

$y \ni \{y \supset \smile\text{‘}((a)^{\times}) : p \,\varepsilon\, ((a)^{\times}) \,.\supset_p.\, y \frown p \,\varepsilon\, 1 : x = \smile\text{‘}y\} \,\varepsilon\, 1,$ (2)

$(1) \,.\, (2) \,.\supset.\, \text{Prop}].$

·2 $\mu\, ((a)^{\times})^{\times} = \mu\, \{\smile\text{‘}((a))\}^{\times},$ $[★ 10{\cdot}12 \,.\, ★ 10{\cdot}14 \,.\supset.\, \text{Prop}].$

·21 $\overset{p\,\epsilon\,((a))}{\prod}\ \overset{q\,\epsilon\,p}{\prod} \mu q = \mu\, ((a)^{\times})^{\times}.$ Df.

·22 $\overset{p\,\epsilon\,((a))}{\prod}\ \overset{q\,\epsilon\,p}{\prod} \mu q = \overset{q\,\epsilon\,\smile\text{‘}((a))}{\prod} \mu q,$ $[★ 10{\cdot}2 \,.=.\, \text{Prop}].$

Note: This proposition, or its equivalent ★ 10·2, is the associative law of multiplication when the number of brackets and the number of factors in each bracket is not necessarily finite or denumerable.

★ 11

·0 $d \,\varepsilon\, \text{cls}^2\,\text{excl} \,.\, f \,\varepsilon\, \text{cls} \,.\, f \frown \smile\text{‘}d = \Lambda \,.\, h = \iota(\smile\text{‘}d) \smile \iota f \,.$

$((g)) = (\iota\iota f \smile \iota d)^{\times} \,.\supset.\, \smile\text{‘}((g)^{\times}) = h$

$[\smile\text{‘}((g)^{\times}) = m \ni \{f \frown m \,\varepsilon\, 1 \,.\, d \frown p \ni (p \frown m \,\varepsilon\, 1) \,\varepsilon\, 1 \,.\, m \supset f \smile (\smile\text{‘}d)\}$

$= m \ni \{f \frown m \,\varepsilon\, 1 \,.\, \smile\text{‘}d \frown m \,\varepsilon\, 1 \,.\, m \supset f \smile (\smile\text{‘}d)\} = h^{\times}].$

Note: This is the distributive law for addition and multiplication; it can be symbolized thus:

·1 $\alpha \,\varepsilon\, Nc \,.\, d \,\varepsilon\, \text{cls}^2\,\text{excl} \,.\supset.\, \alpha \times \mu\, (\smile\text{‘}d) = \overset{p\,\epsilon\,d}{\sum} \alpha \times \mu p.$

SECTION V.

On Powers, Combinations and Permutations.

★12 $\alpha, \beta \,\varepsilon\, Nc \,.\supset \therefore$

·0 $a, b \,\varepsilon\, \alpha \cap \text{cls}'\beta \text{ excl} \,.\supset.\, \mu a^{\times} = \mu b^{\times},$ [★7·2 . $\supset$. Prop].

·1 $\beta^{\alpha} = \text{℩} Nc \cap x \ni \{a \,\varepsilon\, \alpha \cap \text{cls}'\beta \text{ excl} \,.\supset.\, x = \mu a^{\times}\}.$ Df.

★13 $((a)) \,\varepsilon\, \text{cls}^3 \text{arithm} \,.\, \varpi \,\varepsilon\, Nc : q \,\varepsilon^2\, ((a)) \,.\supset.\, q \,\varepsilon\, \varpi : \supset ::$

·0 $\mu\, ((a)^{\times})^{\times} = \overset{p\,\varepsilon((a))}{\prod} \varpi^{\mu p},$ [★12·1 . $\supset$. Prop].

·1 $\gamma \,\varepsilon\, Nc \,.\, \cup{}^{\backprime}((a)) \supset \gamma \,.\supset.\, \overset{p\,\varepsilon((a))}{\prod} \gamma^{\mu p} = \gamma^{\overset{p\,\varepsilon((a))}{\Sigma} \mu p},$ [★10·22 . $\supset$. Prop].

·2 $\alpha, \beta, \gamma \,\varepsilon\, Nc \,.\supset.\, (\gamma^{\beta})^{\alpha} = \gamma^{\alpha \times \beta},$

[Hp (★13·1) . $((a)) \,\varepsilon\, \alpha \cap \text{cls}'\beta$. ★13·1 . ★7·3 . $\supset$. Prop].

★14 $u, v \,\varepsilon\, \text{cls} \,.\, u \cap v = \Lambda \,.\supset :$

·0 $v^{u} = Nc \rightarrow 1 \cap R \ni (\rho = u \,.\, \breve{\rho} \supset v).$ Df.

·1 $\mu\, (v^{u}) = \mu v^{\mu u},$

[$x \,\varepsilon\, u \,.\supset_x.\, w_x = (\iota x \cup \iota v)^{\times} : w = k \ni \{\exists u \cap x \ni (k = w_x)\} : \supset \therefore$

$p \,\varepsilon\, w^{\times} \,.=:\, x \,\varepsilon\, u \,.\supset_x.\, w_x \cap p \,\varepsilon\, 1 : p \supset \cup{}^{\backprime} w \therefore \supset .\, \mu w^{\times} = \mu\, (v^{u})$ (1)

$w \,\varepsilon\, \mu u \cap \text{cls}'\mu v \,.\supset.\, \mu w^{\times} = \mu v^{\mu u}, \ldots .$ (2) (1) . (2) . $\supset$. Prop].

Note: This proposition deduces Cantor's definition of a power [cf. Math. Ann., Bd. XLVI] from the definition of a power given in ★12·1.

★15 $a \,\varepsilon\, \text{cls} \,.\supset \therefore$

·0 $\mu\ \text{cls}'a = 2^{\mu a},$

[$x \,\varepsilon\, a \,.\supset_x.\, a_x = \text{cls}'a \cap u \ni (x \,\varepsilon\, u) \,.\, a'_x = \text{cls}'a - a_x :$

$c = k \ni \{\exists a \cap x \,\varepsilon\, (k = \iota a_x \cup \iota a'_x)\} : \supset .\, \mu c^{\times} = 2^{\mu a},$ (1)

$m \,\varepsilon\, c^{\times} \,.\supset.\, d_m = \text{℩}\,\text{cls}'a \cap u \ni \{x \,\varepsilon\, a \,.\supset \therefore a_x \,\varepsilon\, m \,.\supset.\, x \,\varepsilon\, u : a'_x \,\varepsilon\, m \,.\supset.\, x - \varepsilon\, u\} :$

$\supset .\, \text{cls}'\, a = w \ni \{\exists c^{\times} \cap m \ni (w = d_m)\} \,.\supset.\, \mu\ \text{cls}'a = \mu c^{\times},$ (2)

(1) . (2) . $\supset$. Prop].

·1 $\beta \,\varepsilon\, Nc \,.\supset.\, C^{a}_{\beta} = \text{cls}'a \cap \beta.$ Df.

·2 $\text{cls}'a = x \ni \{\exists Nc \cap \beta \ni (x \,\varepsilon\, C^{a}_{\beta})\}.$

·3 $\overset{\beta \leqq \mu a}{\sum} \mu C^{a}_{\beta} = \mu\ \text{cls}'a = 2^{\mu a}.$

·4 $a' \text{ sim } a \,.\, \beta \,\varepsilon\, Nc \,.\supset.\, \mu C^{a}_{\beta} = \mu C^{a'}_{\beta}.$

★16 $\alpha, \beta, \gamma \,\varepsilon\, Nc \,.\supset ::$

★16·0 $C^a_\beta = \iota Nc \frown x \ni \{a \varepsilon \alpha \,.\supset .\, C^a_\beta \varepsilon x\}$. Df.

·1 $\sum^{\rho+\sigma=\gamma} C^\alpha_\rho \times C^\beta_\sigma = C^{\alpha+\beta}_\gamma,$

$[a \varepsilon \alpha \,.\, b \varepsilon \beta \,.\, a \frown b = \Lambda \,.\, u = \gamma \frown z \ni \{\exists (x, y) \ni (x \supset a \,.\, y \supset b \,.\, z = x \smile y)\} \,.$

$\supset .\, u = \gamma \frown z \ni (z \supset a \smile b) \,.\supset .$ Prop].

·2 $\alpha \varepsilon Nc$ fin $.\supset .\, C^\alpha_\alpha = 1$.

·21 $\alpha \varepsilon Nc$ infin $.\, \rho \varepsilon Nc \,.\, \rho \leqq \alpha \,.\, a \varepsilon \alpha \,.\supset .\, C^\alpha_{\alpha_\rho} = C^\alpha_\alpha \frown p \varepsilon (a - p \varepsilon \rho)$. Df.

Note: The class of infinite numbers denoted in this memoir by $N\alpha_0$ is defined below (cf. ★30·0); it is the class of numbers such that if β be any one of its members, $\beta + \beta = \beta$, and hence if γ be any smaller number $\beta + \gamma = \beta$. In Section III these propositions are deduced by the use of the hypothetical primitive proposition ★4·3; and in the sets of propositions ★30, ★31 and ★32, properties of the class are discussed. It will be noticed that in the proof of ★19·7 there is a forward reference to ★31·3.

·22 $\alpha \varepsilon N\alpha_0 \,.\, \rho \varepsilon Nc \,.\, \rho < \alpha \,.\, a \varepsilon \alpha \,.\supset .\, \mu C^a_{\alpha_\rho} = C^\alpha_\rho,$

[★4·38 $.\supset .\, \alpha - \rho = \alpha \,.\supset .$ Prop].

·23 $\alpha \varepsilon N\alpha_0 \,.\, a \varepsilon \alpha \,.\supset .\, \mu C^a_{\alpha_\alpha} = C^a_\alpha$.

$[\alpha + \alpha = \alpha \,.\supset .\, \exists\, \mathrm{cls}'a \frown \alpha \frown u \ni (a - u \varepsilon \alpha) \,.\supset .\, C^u_\alpha \supset C^a_{\alpha_\alpha} \,.\supset .\, C^a_\alpha \leqq \mu C^a_{\alpha_\alpha},$(1)

$C^a_{\alpha_\alpha} \supset C^a_\alpha \,.\supset .\, \mu C^a_{\alpha_\alpha} \leqq C^a_\alpha,$ .... (2) (1) . (2) $.\supset .$ Prop].

·24 $\alpha \varepsilon N\alpha_0 \,.\supset .\, C^a_\alpha = 2^\alpha,$ $[a \varepsilon \alpha \,.\supset .\, C^a_\alpha = \sum^{\rho \leqq \alpha} \mu C^a_{\alpha_\rho} = \sum^{\rho \leqq \alpha} C^a_\rho = 2^\alpha]$.

·3 $\alpha, \beta \varepsilon Nc \,.\supset .\, \alpha^\beta \leqq C^{\alpha\beta}_\beta,$

$[a \varepsilon \beta \frown \mathrm{cls}'\alpha$ excl $.\supset .\, \mu \smile{}'a = \alpha\beta \,.\, \mu a^\times = \alpha^\beta,$ (1)

$a^\times \supset \beta \,.\supset .\, a^\times \supset C^{\smile 'a}_\beta,$ .... (2) (1) . (2) $\supset$ Prop].

·31 $\alpha \times 2^\beta \leqq C^{\alpha\beta}_\beta$ $[a \varepsilon \alpha \frown \mathrm{cls}'\beta$ excl $.\supset .\, \mu \smile{}'a = \alpha\beta,$ (1)

$b = q \ni \{\exists a \frown p \ni (q = C^p_\beta)\} \,.\supset .\, b \varepsilon \alpha \frown \mathrm{cls}'\, 2^\beta \,.\supset .\, \mu \smile{}'b = \alpha \times 2^\beta,$ (2)

$\smile{}'b \supset C^{\smile 'a}_\beta,$ .... (3) (1) . (2) . (3) $.\supset .$ Prop].

Binomial Theorem.—The next three sets of props (under integral numbers 17 to 19) are devoted to proving forms of the Binomial Theorem. The main results are given in ★17·32 and ★17·4 and ★18·53, ★19·4 and ★19.7.

★17 $\alpha, \beta, \delta, \nu \varepsilon Nc \,.\, R \varepsilon 1 \to 1 \,.\, \rho \varepsilon \alpha \frown \mathrm{cls}'\beta$ excl $.\, \breve{\rho} \varepsilon \alpha \frown \mathrm{cls}'\delta$ excl .

$\smile{}'\rho \frown \smile{}'\breve{\rho} = \Lambda \,.\, a = p \ni \{\exists \rho \frown q \ni (p = q \smile \iota \breve{\rho} q)\} \,.\supset .\therefore$

·0 $a \varepsilon \alpha \frown \mathrm{cls}'(\beta + \delta)$.

·1 $a^\times \varepsilon (\beta + \delta)^\alpha \frown \mathrm{cls}'\alpha$.

★17·11 $m \,\varepsilon\, a^{\times} . q \,\varepsilon\, \rho . \supset . m \cap q \,\varepsilon\, 1 \cup 0.$

·2 $a_{\rho_\nu} = a^{\times} \cap m \ni \{ \cup \text{‘}\rho \cap m \,\varepsilon\, \nu \}$ Df.

·21 $l_\nu \,\varepsilon\, C^\rho_\nu . \supset . \breve{l}_\nu = \breve{\rho} - \breve{\rho}\,(l_\nu).$ Df.

·211 $l_\nu \,\varepsilon\, C^\rho_\nu . \supset . \mu\breve{l}_\nu \,\varepsilon\, \sigma \ni (\sigma + \nu = \alpha).$

·212 $l_\nu \,\varepsilon\, C^\rho_\nu . \supset . \mu\breve{l}^{\times}_\nu = \delta^{\mu\breve{l}_\nu}$

·22 $a_{\rho_\nu} = m \ni [\exists\,(p, q) \ni \{\exists\, C^\rho_\nu \cap l_\nu \ni (p \,\varepsilon\, l^{\times}_\nu . q \,\varepsilon\, l^{\times}_\nu . m = p \cup q)\}].$

·23 $\mu a_{\rho_\nu} = \sum^{l_\nu \varepsilon C^\rho_\nu} \mu l^{\times}_\nu \times \mu l^{\times}_\nu = \sum^{l_\nu \varepsilon C^\rho_\nu} \beta^\nu \times \mu l^{\times}_\nu = \beta^\nu_{\times} \sum^{l_\nu \varepsilon C^\rho_\nu} \delta^{\mu\breve{l}_\nu}.$

·30 $a_\rho = k \ni \{\exists\, Nc \cap \nu \ni (k = a_{\rho_\nu})\}.$ Df·

·31 $a_\rho \,\varepsilon\, \text{cls}^2 \,\text{excl} . \cup \text{‘} a_\rho = a^{\times}.$

·32 $(\beta + \delta)^\alpha = \mu a^{\times} = \mu \cup \text{‘} a_\rho = \sum^{\nu \leqq \alpha} \mu a_{\rho_\nu} = \sum^{\nu \leqq \alpha} \beta^\nu \times \sum^{l_\nu \varepsilon C^\rho_\nu} \delta^{\mu\breve{l}_\nu}$

·4 $\delta = 1 . \supset . (1 + \beta)^\alpha = \sum^{\nu \leqq \alpha} \beta^\nu \times \mu C^\rho_\nu = \sum^{\nu \leqq \alpha} C^\alpha_\nu \times \beta^\nu.$

★18 Hp (★17) . $\alpha \,\varepsilon\, Nc$ fin . $\supset$:

·1 $\nu \leqq \alpha . \supset . \sigma \ni (\sigma + \nu = \alpha) \,\varepsilon\, 1 . u\breve{l}_\nu = \alpha - \nu.$

[The second part of the prop follows from ★17·211].

·2 $\sum^{l_\nu \varepsilon C^\rho_\nu} \delta^{\mu\breve{l}_\nu} = C^\alpha_\nu \times \delta^{\alpha - \nu}.$

·3 $(\beta + \delta)^\alpha = \sum^{\nu \leqq \alpha} C^\alpha_\nu \times \beta^\nu \times \delta^{\alpha - \nu}.$

★19 Hp (★17) . $\alpha \,\varepsilon\, N\alpha_0 . \supset .\!\cdot.$

·0 $\nu < \alpha . \sigma + \nu = \alpha . \supset . \sigma = \alpha.$

·1 $\nu < \alpha . \supset . \sum^{l_\nu \varepsilon C^\rho_\nu} \delta^{\mu\breve{l}_\nu} = C^\alpha_\nu \times \delta^\alpha,$

[★17·211 . ★19·0 . $\supset$. Prop].

·2 $\nu \leqq \alpha . \supset : l_\alpha \,\varepsilon\, C^\rho_{\alpha_\nu} . = . \mu\breve{l}_\alpha = \nu.$

·3 $\sum^{l_\alpha \varepsilon C^\rho_\alpha} \delta^{\mu\breve{l}_\alpha} = \sum^{\nu \leqq \alpha} \mu C^\rho_{\alpha_\nu} \times \delta^\nu = \sum^{\nu \leqq \alpha} C^\alpha_\nu \times \delta^\nu = (1 + \delta)^\alpha.$ [★17·4 . $\supset$. Prop]

·4 $(\beta + \delta)^\alpha = \sum^{\nu < \alpha} C^\alpha_\nu \times \beta^\nu \times \delta^{\alpha - \nu} + \beta^\alpha (1 + \delta)^\alpha,$

[★17·32 . ★19·3 . $\supset$. Prop].

★19·7 $(\beta + \delta)^\alpha = \delta^\alpha \times (1 + \beta)^\alpha + \beta^\alpha \times (1 + \delta)^\alpha,$

$$[(\beta+\delta)^\alpha = \sum^{\nu<\alpha} C^\alpha_\nu \times \beta^\nu \times \delta^{\alpha-\nu} + \beta^\alpha \times (1+\delta)^\alpha$$

$$= \sum^{\nu<\alpha} C^\alpha_\nu \times \delta^\nu \times \beta^{\alpha-\nu} + \delta^\alpha \times (1+\beta)^\alpha, \qquad (1)$$

$$\sum^{\nu-\alpha} C^\alpha_\nu \times \beta^\nu \times \delta^{\alpha-\nu} = \delta^\alpha \sum^{\nu<\alpha} C^\alpha_\nu \times \beta^\nu \leqq \delta^\alpha \sum^{\nu\leqq\alpha} C^\alpha_\nu \times \beta^\nu \leqq \delta^\alpha \times (1+\beta)^\alpha. \qquad (2)$$

$$\sum^{\nu<\alpha} C^\alpha_\nu \times \delta^\nu \times \beta^{\alpha-\nu} \leqq \beta^\alpha (1+\delta)^\alpha, \qquad (3)$$

$$\star 31{\cdot}3 \,.\supset : \beta^\alpha \times (1+\delta)^\alpha + \delta^\alpha \times (1+\beta)^\alpha = \beta^\alpha \times (1+\delta)^\alpha \,.\cup.$$

$$\beta^\alpha \times (1+\delta)^\alpha + \delta^\alpha \times (1+\beta)^\alpha = \delta^\alpha \times (1+\beta)^\alpha, \qquad (4)$$

$(1) . (2) . (3) . (4) . \supset .$ Prop].

★20·1 $\alpha \,\varepsilon\, N\alpha_0 . \beta, \delta \,\varepsilon\, Nc . \beta \geqq \delta . \supset . (\beta \times \delta)^\alpha = \beta^\alpha,$

$[\alpha \,\varepsilon\, N\alpha_0 . \beta \geqq \delta . \supset . \beta^\alpha = \beta^{\alpha+\alpha} = (\beta \times \beta)^\alpha \geqq (\beta \times \delta)^\alpha,$ (1)

$(\beta \times \delta)^\alpha \geqq \beta^\alpha,$ (2)

$(1) . (2) . \supset .$ Prop].

★21 $\alpha \,\varepsilon\, N\alpha_0 . \beta \,\varepsilon\, Nc . \supset .\!\cdot\!.$

·1 $(2\beta)^\alpha = \beta^\alpha (1 + \beta)^\alpha,$ [★19·7 . $\beta = \delta . \supset .$ Prop].

·2 $(1 + \beta)^\alpha = (2\beta)^\alpha + (1 + \beta)^\alpha,$ [★19·7 . $\delta = 1 . \supset .$ Prop].

·3 $(2\beta)^\alpha = (1 + \beta)^\alpha,$ [★21·1 . ★21·2 . $\supset .$ Prop].

·4 $\nu \,\varepsilon\, Nc \text{ fin} . \supset \nu^\alpha = 2^\alpha,$

[$\sigma \,\varepsilon\, Nc$ fin $- \iota 0 . \beta = 2^\sigma + 1$. ★21·3 . Induct . $\supset$.

$$(2^{\sigma+1} + 2)^\alpha = (2^\sigma + 2)^\alpha = (2 + 2)^\alpha = 3^\alpha, \qquad (1)$$

$(2 + 2)^\alpha = 2^{2\alpha} = 2^\alpha,$ (2)

$\nu \,\varepsilon\, Nc$ fin $- \iota 0 . \; 2^{\sigma+1} + 2 \geqq \nu \geqq 2^\sigma + 2 . \supset . (2^{\sigma+1} + 2)^\alpha \geqq \nu^\alpha \geqq (2^\sigma + 2)^\alpha,$ (3)

$(1) . (2) . (3) . \supset .$ Prop].

★22 $\alpha, \beta \,\varepsilon\, Nc . \alpha \geqq \beta . a \,\varepsilon\, \alpha . b \,\varepsilon\, \beta . \supset .\!\cdot\!.$

·0 $P^\alpha_b = 1 \rightarrow 1 \cap R \ni (\rho = b . \breve{\rho} \supset a).$ Df.

·01 $P^\alpha_\beta = \text{ɿ} Nc \cap \gamma \ni (a \,\varepsilon\, \alpha . b \,\varepsilon\, \beta . \supset . P^\alpha_b \ni \gamma).$ Df.

·02 $P^\alpha_b \supset a^b,$ [★14·0 . ★22·0 . $\supset$. Prop].

·03 $P^\alpha_\beta \leqq \alpha^\beta,$ [★14·1 . ★22·02 . $\supset$. Prop].

·04 ${}^\beta P_\beta = \text{ɿ} Nc \cap \gamma \ni \{1 \rightarrow 1 \cap R \ni (\rho = b . \breve{\rho} = b) \,\varepsilon\, \gamma\}.$ Df.

·05 ${}^\beta P_\beta \leqq P^\beta_\beta.$

·06 $\beta \,\varepsilon\, Nc \text{ fin} . \supset . {}^\beta P_\beta = P^\beta_\beta.$

★22·1 $^{\beta}P_{\beta}\times C^{\alpha}_{\beta}=P^{\alpha}_{\beta},$

$[a\,\varepsilon\,\alpha\,.\,b\,\varepsilon\,\beta:l\,\varepsilon\,C^{\alpha}_{\beta}\,.\supset_{l}.\,k_{l}=1\rightarrow 1\cap R\ni(\rho=b\,.\,\breve{\rho}=l):$

$k=p\ni\{\exists\,C^{\alpha}_{\beta}\cap l\ni(p=k_{l})\}:\supset.\,\mu k=\mu C^{\alpha}_{\beta}=C^{\alpha}_{\beta}\,.$

$k\supset{}^{\beta}P_{\beta}\,.\cup\text{‘}k=\beta^{\alpha}_{b}:\supset P_{\beta}=\mu P^{\alpha}_{b}=\mu\cup\text{‘}k={}^{\beta}P_{\beta}\times C^{\alpha}_{\beta}].$

·2 $\alpha\,\varepsilon\,N\alpha_{0}\,.\,\beta\,\varepsilon\,Nc\ \mathrm{fin}\,.\cup.\,\beta\,\varepsilon\,N\alpha_{0}\,.\,\alpha\geqq\beta:\supset.\,C^{\alpha\times\beta}_{\beta}=P^{\alpha\times\beta}_{\beta}=\alpha^{\beta},$

$[\star 16{\cdot}3\,.\star 22\,.\,03\,.\star 22{\cdot}1\,.\supset.\,\alpha^{\beta}\leqq C^{\alpha\times\beta}_{\beta}\leqq P^{\alpha\times\beta}_{\beta}\leqq(\alpha\times\beta)^{\beta},$ (1)

$\mathrm{Hp}\,.\star 4{\cdot}42\,.\star 20{\cdot}1\,.\supset.\,\alpha^{\beta}=(\alpha\times\beta)^{\beta},$ .... (2) $(1)\,.\,(2)\,.\supset.\,\mathrm{Prop}],$

·21 $\alpha\,\varepsilon\,N\alpha_{0}\,.\,\beta\,\varepsilon\,Nc\cap\beta'\ni(\alpha\times\beta'=\alpha)\,.\supset.\,C^{\alpha}_{\beta}=P^{\alpha}_{\beta}=\alpha^{\beta},$

$[\star 22{\cdot}2\,.\supset.\,\mathrm{Prop}].$

·22 $\alpha\,\varepsilon\,N\alpha_{0}\,.\,\alpha^{2}=\alpha\,.\supset.\,\alpha^{\alpha}=2^{\alpha},$ $[C^{\alpha}_{\alpha}=2^{\alpha}\,.\star 22{\cdot}21\,.\supset.\,\mathrm{Prop}].$

★23·1 $\beta\,\varepsilon\,Nc\,.\supset.\,\beta<2^{\beta},$

$[a\,\varepsilon\,\beta\cap\mathrm{cls'}\,2\ \mathrm{excl}\,.\,q,\,q'\,\varepsilon\,a^{\times}.\,q\cap q'=\Lambda:p\,\varepsilon\,a\,.\supset_{p}.$

$p'=\imath a^{\times}\cap m\ni(m=q-p\cup q'\cap p):b=x\ni\{\exists a\cap p\ni(x=p')\}:\supset.$

$b\supset a^{\times}.\,\mu b=\mu a:\supset.\,ma^{\times}\geqq\mu a,$ (1)

$\alpha\,\varepsilon\,\beta\cap\mathrm{cls'}\,2\ \mathrm{excl}\,.\,S\,\varepsilon\,1\rightarrow 1\,.\,\sigma=a\,.\,\breve{\sigma}\supset a^{\times}:p\,\varepsilon\,a\,.\supset.\,p_{1}=\imath(p\cap\iota\breve{\sigma}p)\,.$

$p_{2}=\imath(p-\iota p_{1}):q=x\ni\{\exists a\cap p\ni(x1\text{'}p_{2})\}:\supset.\because$

$q\,\varepsilon\,a^{\times}:p\,\varepsilon\,a\,.\supset_{p}.\,p_{2}\,\varepsilon\,q-\breve{\sigma}p\,.\because.\supset.\,q\,\varepsilon\,a^{\times}-\breve{\sigma}a,$ (2)

$(1)\,.\,(2)\,.\supset.\,\mathrm{Prop}].$

Note: This proposition is the extension for any cardinal number of Cantor's proof [cf. Jahresber. d. Deutsch. Math. Ver. I, S. 77] that the power of the continuum (2^{α_0}) is greater than the first infinite cardinal (α_0). The adaptation of the proof and its reduction to symbols are due to Russell.

In Section IV and the present Section V, all the proofs involving any characteristic properties of the class *Nc* infin have been limited to the class $N\alpha_0$ and based on the theorem that

$$\alpha\,\varepsilon\,N\alpha_{0}\,.\,\beta\leqq\alpha\,.\supset.\,\alpha+\beta=\alpha.$$

It becomes important therefore to investigate for what class contained within *Nc*, the theorem

$$\beta\leqq\alpha\,.\supset.\,\alpha+\beta=\alpha$$

can be asserted, since the universal validity of ★4·3 is not proved. Furthermore, we have not succeeded in deducing, even from ★4·3, the allied proposition

$$\alpha\,\varepsilon\,Nc\ \mathrm{infin}\,.\,\beta\leqq\alpha\,.\supset.\,\alpha\times\beta=\alpha.$$

The nearest approach to it as yet is ★20·1.

The sets of propositions with integral numbers 30 to 32 will be devoted to these investigations; the proofs of some propositions practically proved by Russell in Section III will be omitted. In these sets of propositions (30 to 32) it will be found that the only properties of an Nc infin used are that [cf. ★4·1]

$$\alpha\ \varepsilon\ Nc\ \text{infin}\,.\supset.\,\alpha \geqq \alpha_0\,.\,\alpha+\alpha_0=\alpha\,,$$

where α_0 is the first cardinal number.

★30

·0 $N\alpha_0 = Nc \frown \beta \ni [\exists Nc \frown \delta \ni (\delta > 0\,.\,\beta = \alpha_0 \times \delta)]\,.$ Df.

Note: As long as all the numbers considered belong to $N\alpha_0$, the propositions in Section III, ★4·31 to 4·51, hold without modification.

·1 $\alpha\ \varepsilon\ N\alpha_0\,.\,\beta\ \varepsilon\ Nc\,.\supset\therefore\alpha \geqq \beta\,.\supset.\,\alpha+\beta=\alpha : \alpha \leqq \beta\,.\supset.\,\alpha+\beta=\beta,$

$[\alpha \geqq \beta\,.\supset.\,\alpha \leqq \alpha+\beta \leqq \alpha+\alpha \leqq \alpha\,.\supset.\,\alpha+\beta=\alpha,$ (1)

$\alpha \leqq \beta\,.\supset.\,\exists Nc \frown \gamma \ni (\alpha+\gamma=\beta)\,.\supset.\,\alpha+\beta=\alpha+\alpha+\gamma=\alpha+\gamma=\beta,$ (2)

$(1)\,.\,(2)\,.\supset.\,\text{Prop}].$

·2 $\beta, \gamma\ \varepsilon\ Nc\,.\,\exists N\alpha_0 \frown \alpha \ni [\beta \leqq \alpha \leqq \gamma]\,.\supset.\,\beta+\gamma=\gamma,$

$[\gamma \leqq \beta+\gamma \leqq \alpha+\gamma \leqq \gamma\,.\supset.\,\text{Prop}].$

Note: This theorem is important, and enables the enunciations of some theorems of ★21 and ★22 to be extended.

·3 $\beta\ \varepsilon\ Nc\,.\,\alpha\varepsilon\ N\alpha_0 \frown z \ni (z \geqq \beta)\,.\supset.\,\alpha \geqq \alpha_0 \times \beta,$

$[\alpha \geqq \beta\,.\supset.\,\alpha_0 \times \alpha \geqq \alpha_0 \times \beta\,.\supset.\,\alpha \geqq \alpha_0 \times \beta].$

·4 $\beta\ \varepsilon\ Nc\,.\,\gamma\ \varepsilon\ Nc \frown z \ni (\beta \leqq \gamma \leqq \alpha_0 \times \beta)\,.\supset.\,\alpha_0 \times \beta = \alpha_0 \times \gamma,$

$[\bigstar 30{\cdot}3\,.\supset.\,\text{Prop}].$

·5 $\alpha, \alpha'\ \varepsilon\ N\alpha_0\,.\,\alpha < \alpha'\,.\,\beta\ \varepsilon\ Nc \frown \beta \ni (\alpha = \alpha_0 \times \beta)\,.$

$\beta'\ \varepsilon\ Nc \frown \beta' \ni (\alpha' = \alpha_0 \times \beta')\,.\supset.\,\beta \leqq \alpha_0 \times \beta < \beta' \leqq \alpha_0 \times \beta',$

$[\bigstar 30{\cdot}4\,.\supset.\,\text{Prop}].$

★31

·1 $Nc\ \text{exp} = Nc \frown \alpha \ni [\exists Nc\ \text{infin} \frown \beta \ni \{\exists Nc \frown \gamma \ni (\gamma > 1\,.\,\alpha = \gamma^{\beta})\}]\,.$ Df.

·11 $u\ \varepsilon\ \text{cls infin}\,.\supset.\,\mu\ \text{cls}'u\ \varepsilon\ Nc\ \text{exp},$

$[\bigstar 15{\cdot}0\,.\supset.\,\text{Prop}].$

·2 $\alpha\ \varepsilon\ Nc\ \text{exp}\,.\supset.\,\alpha > \alpha_0.$

·3 $Nc\ \text{exp}\,.\supset.\,N\alpha_0,$

$[\gamma^{\beta}\ \varepsilon\ Nc\ \text{exp}\,.\supset.\,\gamma > 1\,.\supset.\,\gamma^{\alpha_0}\varepsilon\ Nc\ \text{exp}\,.\supset.\,\gamma^{\alpha_0} > \alpha_0,$ (1)

$(1)\,.\supset.\,\gamma^{\beta} \leqq \alpha_0 \times \gamma^{\beta} \leqq \gamma^{\alpha_0} \times \gamma^{\beta} \leqq \gamma^{\alpha_0+\beta},$ (2)

$\gamma^{\beta}\ \varepsilon\ Nc\ \text{exp}\,.\supset.\,\beta\ \varepsilon\ Nc\ \text{infin}\,.\supset.\,\alpha_0+\beta=\beta,$ (3)

$(2)\,.\,(3)\,.\supset.\,\gamma^{\beta} = \alpha_0 \times \gamma^{\beta}\,.\supset.\,\text{Prop}].$

★32

·1 $Nc \exp^{\alpha_0} = N\alpha_0 \cap \alpha \ni [\exists N\alpha_0 \cap \beta \ni \{\exists Nc \cap \gamma \ni (\gamma > 1 \,.\, \alpha = \gamma^{\beta})\}]$, Df.

·2 $\beta \,\varepsilon\, Nc \exp^{\alpha_0} . \supset . \beta^2 = \beta$,
$[\beta = \gamma^{\delta} . \supset . \beta^2 = \gamma^{2\delta} = \gamma^{\delta} = \beta]$.

·3 $\beta \,\varepsilon\, Nc \,.\, \gamma \,\varepsilon\, Nc \exp^{\alpha_0} . \beta \leqq \gamma \,.\supset.\, \beta\gamma = \gamma$.

·4 $\alpha \,\varepsilon\, Nc \exp^{\alpha_0} . \beta \,\varepsilon\, Nc - \iota 0 - \iota 1 \,.\, \beta \leqq 2^{\alpha} . \supset . \beta^{\alpha} = 2^{\alpha}$,
$[2^{\alpha} \leqq \beta^{\alpha} \ldots\ldots (1)$, Hp . ★32·2 $. \supset . \beta^{\alpha} \leqq (2^{\alpha})^{\alpha} \leqq 2^{\alpha^2} \leqq 2^{\alpha}$, (2)
(1) . (2) $. \supset .$ Prop].

·5 $\delta \,\varepsilon\, Nc \exp^{\alpha_0} . \gamma \,\varepsilon\, Nc - \iota 0 - \iota 1 \,.\, \alpha \,\varepsilon\, Nc \exp^{\alpha_0} . \delta \geqq \alpha \,.\, \beta = \gamma^{\delta} . \supset . \beta^{\alpha} = \beta$,
$[\beta^{\alpha} = \gamma^{\alpha \times \delta} = \gamma^{\delta} = \beta]$.